MATH202 and MATH291

Analytic Topic

Produced By

Dr Graeme Morris

and

Assoc/Prof Annette L. Worthy
MATH202 Differential Equations II comprises of two sections, namely,

Analytic Solutions to Differential Equations (or DE Section) and

Numerical Methods.

In the Analytic Section, techniques are used to solve particular types of differential equations and some partial differential equations.

The notes and tutorial exercises themselves do not form a complete set for this section of MATH202, and are only intended as a summary of the more comprehensive material to be presented in lectures. They are not intended as a substitute for attending lectures and it is not possible to learn the Differential Equation section of MATH202 by using these notes alone. Students will be expected to use other reference material to extend their knowledge of this section.

The problems are intended to be a minimum set for survival. Students should attempt all of the exercises and are encouraged to seek out additional problems and work from one or more references, or other books in the library.

Apart from the included tutorial exercises, students should seek additional work from one or more of the references, or other books in the library. Extra problems and reading are especially recommended for those students who are struggling or who wish to extend themselves and gain more than just a pass. However, students who are struggling in the Differential Equation section should seek advice immediately by contacting the lecturer concern.

Any further information regarding the Analytic Section of MATH202 can be obtained from your the MATH202 subject co-ordinator and/or topic lecturer or from Dr Annette Worthy
School of Mathematics and Applied Statistics
University of Wollongong

Room 15.145
Phone: 02-42-21-3838
email: annette_worthy@uow.edu.au
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# Numerical Methods

## Chapter 9

### 9.1 Introduction to Numerical Integration
of Simple Differential Equations

### 9.2 Runge Kutta Methods

### 9.3 Implicit Methods

### 9.4 Linear Multistep Methods

## Chapter 10

### Exam Papers

- 1999 MATH283 Multiple Choice Test
- 1999 MATH283 Mid-Session Test - Sample
- 1998 MATH202 Exam Paper: Analytic
- 1999 MATH202 Exam Paper + Sample Numerical Questions
- 2000 MATH283 Exam Paper
- 2001 MATH202 Numerical Methods Specimen Questions

## References

## Solutions

## Tables

- Table of Integrals
- Table of Laplace Transforms
Chapter 1: Partial Differentiation

1.1 SINGLE VARIABLE FUNCTIONS

1.1.1 Trigonometric Functions

The exponential form of the sin and cos functions are defined as
\[ \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \]

The trigonometric identity is
\[ \cos^2 x + \sin^2 x = 1. \]

Addition of Angles Properties
\[ \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \]
\[ \sin(A \pm B) = \sinh A \cos B \pm \cos A \sin B \]

Further properties can be found in Section 6.2.4.

1.1.2 Hyperbolic Functions

The hyperbolic sinh and cosh functions are defined as
\[ \cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2}. \]

The hyperbolic identity is
\[ \cosh^2 x - \sinh^2 x = 1. \]

Addition of Angles Properties
\[ \cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B \]
\[ \sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B \]

Also, the inverse hyperbolic functions are:
\[ \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } |x| > 1 \]
\[ \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \]
\[ \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \quad \text{for } |x| < 1 \]
\[ \coth^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \quad \text{for } |x| > 1. \]
1.1.3 Elementary Differentiation

Let \( y = f(x) \) where \( f \) is a differentiable function, then

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.
\]

1.1.4 Differentiation using the Chain Rule

If \( y = f(u) \) and \( u = g(x) \) then

\[
\frac{dy}{dx} = \frac{df}{du} \frac{du}{dx} = f'(u) \frac{du}{dx}.
\]

1.1.5 Implicit Differentiation

Sometimes we cannot express \( y \) in terms of \( x \) explicitly. Therefore, in order to find the derivative of functions defined by equations involving \( x \) and \( y \) we can differentiate term by term by using the chain rule. This procedure is called implicit differentiation.

Example

Consider \( \sin y^2 + e^{-2x^4}y^3 - y = 3 \), find \( \frac{dy}{dx} \).

Method

From the given equation, it can be assumed that \( y = f(x) \) for at least one differentiable function \( f(x) \). We can now find \( \frac{dy}{dx} \) without finding \( f(x) \). Differentiate both sides of the given equation with respect to \( x \) noting that

\[
\frac{d}{dx} \sin y^2 = \frac{d}{dy} \sin y^2 \times \frac{dy}{dx} \quad \text{(using the chain rule)}
\]

\[
\frac{d}{dx} e^{-2x^4}y^3 = -8x^3 e^{-2x^4}y^3 + e^{-2x^4} \frac{d}{dx} y^3 \quad \text{(using the product rule)}
\]

\[
= -8x^3 e^{-2x^4}y^3 + e^{-2x^4} \frac{d}{dy} y^3 \times \frac{dy}{dx} \quad \text{(using the chain rule)}
\]

\[
= -8x^3 e^{-2x^4}y^3 + 3e^{-2x^4} y^2 \frac{dy}{dx}
\]

\[
\frac{d}{dx} y = \frac{dy}{dx} \quad \text{and} \quad \frac{d}{dx} 3 = 0.
\]

Hence, the derivation of the equation with respect to \( x \) becomes

\[
2y \cos y^2 \frac{dy}{dx} - 8x^3 e^{-2x^4} y^3 + 3e^{-2x^4} y^2 \frac{dy}{dx} - \frac{dy}{dx} = 0.
\]

Rearranging we find that

\[
\frac{dy}{dx} = \frac{8x^3 e^{-2x^4} y^3}{2y \cos y^2 + 3e^{-2x^4} y^2 - 1}.
\]

We will use our knowledge of functions of a single variable to find the derivatives of a function of several variables.
1.2 FUNCTIONS OF SEVERAL VARIABLES

A function \( z = f(x, y) \) of two variables is a rule that assigns for each point, \((x, y)\) in a region \(D\) of the plane, a unique value \(z\).

1.2.1 First Order Partial Differentiation

Given \( z = f(x, y) \), then the first order partial derivative of \( z \) with respect to \( x \) is defined as:

\[
\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.
\]

1. \( \frac{\partial z}{\partial x} \) is measuring the gradient (or change) of \( z \) in the \( x \) direction only.

2. \( y \) remains unchanged in the \( x \) direction or \( y \) is constant (a number) in the \( x \) direction.

3. \( \frac{\partial z}{\partial x} \) is the first partial differentiation with respect to \( x \) and can be labelled in many ways such as \( z_x, \frac{\partial f}{\partial x}, f_x, D_xz \) or \( D_x f(x, y) \).

4. Similarly, \( \frac{\partial z}{\partial y} \) is the first partial differentiation with respect to \( y \) and is defined by

\[
\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.
\]

Example:

Let \( z = f(x, y) = xy \). Find \( \frac{\partial z}{\partial x} \) and check using first principles.

Method:

We keep \( y \) constant and differentiate \( f \) with respect to \( x \) in the normal way. Now as \( \frac{dx}{dx} = 1 \),

\[
\frac{\partial z}{\partial x} = 1 \times y = y.
\]

By first principles, recall that

\[
\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.
\]

If \( f(x, y) = xy \) then

\[
f(x + \Delta x, y) = (x + \Delta x)y = xy + y\Delta x.
\]

Therefore,

\[
\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{xy + y\Delta x - xy}{\Delta x}.
\]
So,

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = y.$$  

Hence, the partial differentiation with respect to \(x\) is simply ordinary differentiation with respect to \(x\) whilst holding \(y\) fixed.

To find \(\frac{\partial z}{\partial y}\).

We keep \(x\) constant and differentiate \(f\) with respect to \(y\) in the normal way. That is,

$$\frac{\partial z}{\partial y} = x \times 1 = x.$$  

Hence, the partial differentiation with respect to \(y\) is simply ordinary differentiation with respect to \(y\) whilst holding \(x\) fixed.

### 1.2.2 Rules for Partial Differentiation

(i) If \(z = f(u)\) and \(u = \phi(x, y)\), then

$$\frac{\partial z}{\partial x} = \frac{df}{du} \times \frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x}.$$  

Called the Chain rule.

(ii) If \(z = f(x, y)\) and \(x = s(t)\) and \(y = r(t)\), then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$  

Examples:

(a) Let \(z = x \sin y\) then \(\frac{\partial z}{\partial x} = \sin y\) and \(\frac{\partial z}{\partial y} = x \cos y.\)

(b) \(z = x^2 + y^2\) then \(\frac{\partial z}{\partial x} = 2x\) , \(\frac{\partial z}{\partial y} = 2y.\)

(c) If \(z = f(u)\) and \(u = u(x, y) = xy\), find \(\frac{\partial z}{\partial x}\) and \(\frac{\partial z}{\partial y}\).

Using the Chain rule we have that

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \times \frac{\partial u}{\partial x} = f'(u) \frac{\partial u}{\partial x} = yf'(u).$$

Also, using the Chain rule we have that

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \times \frac{\partial u}{\partial y} = f'(u) \frac{\partial u}{\partial y} = xf'(u).$$
(d) Let \( z = \sin(xy) \) then \( \frac{\partial z}{\partial x} = y \cos(xy) \) and \( \frac{\partial z}{\partial y} = x \cos(xy) \).

(e) Let \( w = z^2 \) and \( z = f(x,y) \).

Recall the chain rule (or rule (i)). That is,
\[
\frac{\partial w}{\partial x} = \frac{dw}{dz} \times \frac{\partial z}{\partial x}.
\]
Then
\[
\frac{\partial w}{\partial x} = 2z \frac{\partial z}{\partial x},
\]
or
\[
\frac{\partial w}{\partial x} = 2zz_x.
\]
That is,
\[
\frac{\partial z^2}{\partial x} = 2zz_x.
\]

(f) Let \( z = f(x,y), \ x = t^2 \) and \( y = e^t \) then
\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
= 2tf_x + e^tf_y
\]
where \( \frac{dx}{dt} = 2t \) and \( \frac{dy}{dt} = e^t \).

(iii) Implicit Differentiation

Given an equation of the form \( g(x,y,z) = 0 \) then it is implied that \( z = f(x,y) \). That is, \( z \) is an implicit function of \( x \) and \( y \).

To find \( \frac{\partial z}{\partial x} \) or \( \frac{\partial z}{\partial y} \) we simply use implicit differentiation similar to that used in the single variable case.

Examples:

(a) Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) where \( z \) is implicitly defined in the equation \( z^2 - x \sin y = 2 \).

Method

Treating the \( y \) as a constant with respect to \( x \), we implicitly differentiate with respect to \( x \) the equation
\[
z^2 - x \sin y = 2. \quad (\star)
\]
Recall that
\[
\frac{\partial}{\partial x}(z^2) = 2zz_x
\]
then implicitly partially differentiating (\star) with respect to \( x \) we find that
\[
2zz_x - \sin y = 0.
\]
Rewriting, we have that
\[ z_x = \frac{\sin y}{2z}. \]

Similarly, upon implicitly partially differentiating (*) with respect to \( y \), we have that
\[ 2z z_y - x \cos y = 0. \]

That is,
\[ z_y = \frac{x \cos y}{2z}. \]

(b) Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) where \( z \) is implicitly defined in the equation
\[ \ln z + z^3 = e^{xy}. \]

Method

Using the Chain rule we find that
\[ \frac{\partial}{\partial x} \ln z = \frac{1}{z} z_x \]
and
\[ \frac{\partial}{\partial y} \ln z = \frac{1}{z} z_y. \]

Also,
\[ \frac{\partial}{\partial x} z^3 = 3z^2 z_x \]
and
\[ \frac{\partial}{\partial y} z^3 = 3z^2 z_y. \]

Therefore, upon partially differentiating \( \ln z + z^3 = e^{xy} \) with respect to \( x \) we have
\[ \frac{1}{z} z_x + 3z^2 z_x = ye^{xy}. \]

That is,
\[ z_x = \frac{ye^{xy}}{\frac{1}{z} + 3z^2} \]
\[ = \frac{y e^{xy}}{1 + 3z^3}. \]

Similarly,
\[ \frac{1}{z} z_y + 3z^2 z_y = xe^{xy}. \]

That is,
\[ z_y = \frac{xe^{xy}}{\frac{1}{z} + 3z^2} \]
\[ = \frac{x e^{xy}}{1 + 3z^3}. \]
Exercise 1A

1 Find \( \frac{dy}{dx} \) for the following functions
   (a) \( y = \ln(\sin x) \)
   (b) \( y = \tan u \) where \( u = e^{x^2} \),
   (c) \( y = f(v) \), where \( v = \frac{1}{x^2 + 1} \)
   (d) \( y^2 + x^2 = \ln(x + y) \).

2 Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) given the following functions of \( z = f(x, y) \).
   (a) \( z = 4x^3y \)
   (b) \( z = x^2 - 3y^3 + xy + 2 \)
   (c) \( z = \frac{1}{2}e^{x^2y^2} \)
   (d) \( z = \sin(x^3y^2) \)
   (e) \( z = e^{xy} \cos x^2 \)

3 Given the following functions, find \( f_x(x, y) \) and \( f_y(x, y) \).
   (a) \( f(x, y) = \frac{2x + y}{x - 2y} \)

(b) \( f(x, y) = x^2ye^{xy} \)
(c) \( f(x, y) = r \) where \( r = \sqrt{x^2 + y^2} \)

4 Show that \( u(x, y) \) and \( v(x, y) \) satisfy the following equations (called the Cauchy-Riemann equations):
   \[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]
   (a) \( u = x^2 - y^2; \ v = 2xy \)
   (b) \( u = e^x \cos y; \ v = e^x \sin y \)
   (c) \( u = \ln(x^2 + y^2); \ v = 2\tan^{-1}\left(\frac{y}{x}\right) \)
   (d) \( u = x^3 - 3xy^2; \ v = 3x^2y - y^3 \)

5 Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) for the following functions
   (you may need to use implicit differentiation)
   (a) \( z = f(u) \), where \( u = \sqrt{x^2 + y^2} \)
   (b) \( z^2 = \cos 2u \), where \( u = \ln(x) + y \)
   (c) \( \sin z^2 + e^u = 0 \), where \( u = \tan^{-1}(x - y) \)
   (d) \( z = x^y \).

1.2.3 A Geometrical Interpretation of \( z = f(x, y) \)

Given \( z = f(x, y) \), this defines a surface in 3D.

The equation of the surface can be written as
\[ \phi(x, y, z) = f(x, y) - z = 0. \]

The normal to this surface is then given by
\[ \nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = (f_x, f_y, f_z) = (z_x, z_y, -1). \]

If the tangent to the surface at a point \((x_0, y_0, z_0)\) has direction ratios \((P, Q, R)\), then
\[ (P, Q, R) \cdot \nabla \phi = 0, \quad \text{where} \quad \nabla \phi = \text{normal}. \]
That is, 
\[ P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R. \tag{*} \]

Equation \((*)\) is known as a first order linear partial differential equation. Hence, given a first order linear partial differential equation of the form \((*)\) the solution is an integral surface of the form \( z = f(x, y) \).

### 1.2.4 Second Order and Higher Partial Differentiation

Let \( z = f(x, y) \) be a twice differentiable function then we can form second order partial derivatives. That is, the second partial derivative of \( z \) with respect to \( x \) and the second partial derivative of \( z \) with respect to \( y \) are respectively,

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right), \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right).
\]

Also, we can find a mixed second order derivative. This is

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)
\]

**Note:** that for most functions \( \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \).

**Notation**

Let \( z = f(x, y) \) then

\[
\frac{\partial^p z}{\partial x^m \partial y^n}
\]

is called the \( p \)th partial derivative of \( z \) where \( m \) of the derivatives are with respect to \( x \) and \( n \) of them are with respect to \( y \). Note here that \( p = m + n \).

**Example**

Find all second order partial derivative terms for \( z = \cos(xy^2) \).

**Method**

Now

\[
\frac{\partial z}{\partial x} = -y^2 \sin(xy^2) \quad \text{and} \quad \frac{\partial z}{\partial y} = -2xy \sin(xy^2).
\]

Therefore,

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right)
= \frac{\partial}{\partial x} \left( -y^2 \sin(xy^2) \right)
= -y^4 \cos(xy^2)
\]

Also,

\[
\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right)
= \frac{\partial}{\partial y} \left( -2xy \sin(xy^2) \right)
= -2x \sin(xy^2) - 4x^2y^2 \cos(xy^2)
\]
and
\[
\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right)
= \frac{\partial}{\partial y} \left( -y^2 \sin(xy^2) \right)
= -2y \sin(xy^2) - 2y^3 x \cos(xy^2).
\]

**Exercise 1B**

1. Find \(\frac{\partial^2 z}{\partial x^2}\), \(\frac{\partial^2 z}{\partial x \partial y}\) and \(\frac{\partial^2 z}{\partial y^2}\) for each of the functions in Exercise 1A, Question 2 and 3.

2. Let \(\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\) (the Laplace operator). Show that \(\nabla^2 u = 0\) \(\nabla^2 v = 0\) for each of the functions in Exercise 1A, Question 4.

3. Find all first and second partial derivatives of \(z\) where
   (a) \(z = \sqrt{x^2 + y^2}\) at \((-1, 2)\)
   (b) \(z = f(2x + 3y) + g(3x - 4y)\) at \((1, 1)\)
   (c) \(z = f(u)\), where \(u = \frac{1}{x^2 + y^2}\)
   (d) \(z = \ln(2u + v)\).

4. Find all first and second partial derivatives.
   (a) \(f(x, y) = x^2 e^{-y}\)
   (b) \(\rho = \sin \phi \cos \theta\)
   (c) \(g(u, v) = \sqrt{u^2 - 3v}\)

### 1.3 TOTAL DIFFERENTIATION

#### 1.3.1 Increments

Consider the graph of the function \(z = f(x, y)\) given below.

Suppose we move, in the \(xy\) plane, from the point \((x, y)\) to another point \((x_0, y_0)\) close by. Then the distances moved in the \(x\) and \(y\) directions are \(\Delta x\) and \(\Delta y\) respectively. Therefore, this new point is

\[(x + \Delta x, y + \Delta y)\]

As a result there is a change \(z\). This change in \(z\) which is represented by \(\Delta z\) can easily be found.

That is, if \(x\) and \(y\) are incremented by \(\Delta x\) and \(\Delta y\) respectively, then the increment or change in \(z\) will be given by

\[\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)\]

Generally, given \(z = f(x, y)\) then

\[\Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + A \Delta x + B \Delta y\]

where the co-efficients \(A\) and \(B\) are functions of \(\Delta x\) and \(\Delta y\).
For small values of $\Delta x$ and $\Delta y$ we find that

$$\Delta z \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y.$$ 

1.3.2 Total Differentials

If the increments $\Delta x$ and $\Delta y$ are relatively “small” then $A\Delta x$ and $B\Delta y$ would be “very small”. Under these conditions, a good approximation to $\Delta z$ can be obtained by discarding these very small terms. Therefore, the approximation to $\Delta z$ is then called the total differentiation of $z$ denoted by $dz$. That is,

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$ 

Since $z = f(x, y)$ then the total differentiation of $z$ can be written as:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

where $dx$ and $dy$ are called the differentials.

Example

(a) Let $z = x^2 + xy$.

Method

The increment in $z$ (or $\Delta z$) is defined as

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = (2x + y)\Delta x + x\Delta y + 2(\Delta x)^2 + \Delta x\Delta y.$$ 

Therefore,

$$\Delta z \approx (2x + y)\Delta x + x\Delta y.$$ 

The total differential, $dz$, is defined as

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (2x + y)dx + xdy.$$ 

Hence, if $\Delta x$ and $\Delta y$ are relatively small we have that

$$\Delta z \approx dz.$$ 

(b) Given $dz = (3x^2y^2 + 2xy)dx + (2x^3y + x^2 + 6)dy$, find $z = f(x, y)$.

Method

Recall that $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ then equating co-efficients of $dx$ we have that

$$\frac{\partial f}{\partial x} = 3x^2y^2 + 2xy.$$ 

Upon integrating with respect to $x$ we find that

$$f(x, y) = x^3y^2 + x^2y + g(y).$$ 

Note: Here that upon the integration with respect to $x$, the constant of integration is $g(y)$. This is due to the fact that $f$ is a function of two variables and therefore, any function of $y$ is constant with respect to $x$. 

We need now find \( g(y) \). This can be done by differentiating \( f(x, y) \) with respect to \( y \) and then equating with the co-efficient of \( dy \). That is,
\[
\frac{\partial f}{\partial y} = 2x^3y + x^2 + g'(y)
\]
and upon equation with the co-efficient of \( dy \) we have that
\[
2x^3y + x^2 + g'(y) = 2x^3y + x^2 + 6.
\]
Solving for \( g'(y) \), we have that
\[
g'(y) = 6.
\]
Upon solving this first order differential equation we have that
\[
g(y) = 6y + c.
\]
Here \( c \) is a constant not dependent on any variable. This is due to the function \( g \) being dependent on one variable only. Therefore,
\[
f(x, y) = x^3y^2 + x^2y + 6y + c.
\]

### 1.3.2 Second Total Differentiation

If \( z = f(x, y) \) then
\[
d^2z = \left( \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^2 z.
\]

#### Exercise 1C

1. Find the increment and the first and second total differentials of \( z \) when
\[
z = x^2 - 3xy + 2y^2
\]
at \( x = 2, y = -3 \) and \( \Delta x = -0.3, \Delta y = 0.2 \).

2. Find the total differential, \( dz \) for each of the following.
   (a) \( z = xy \)  
   (b) \( z = x^y \)
   (c) \( z = \ln(xy) \)  
   (d) \( z = \cos x \sin y \)
   (e) \( z = y \tan x \)

3. Use differentials to estimate \( \sqrt{27} \sqrt{1021} \).
   Compare the estimate to the value obtained using a calculator.

4. The power of an aircraft engine is proportional to the square of its length and the cube of the speed of an aeroplane model. What percentage increase in speed will result from a 13% increase in power and a 2% increase in length of the aircraft.

5. A horizontal beam is supported at both ends and supports a uniform load. The deflection or sag at its middle is given by
\[
S = \frac{k}{wh^3}
\]
where \( w \) and \( h \) are the width and height of the beam, respectively, and \( k \) is a constant that depends on the length and composition of the beam and the amount of the load. If \( S \) is 2.5cms when \( w \) is 5cms and \( h \) is 10cms, approximate the sag when \( w \) is 5.1cms and \( h \) is 10.1cms.

Compare your approximation with the actual value you compute from the given formula for \( S \).

continued next page...
The reactance $X$ of an electric circuit containing the inductance $L$ and the capacitance $C$ is given by the formula

$$X = \omega L - \frac{1}{\omega C}$$

where $\omega$ is the circular frequency of an alternating current. The value of the reactance, $X$ was obtained for $C = 10^{-4}$ farads, $L = 0.2$ and $\omega = 314$ units.

Calculate the change in reactance if the values of the inductance and capacitance are respectively changes to $0.205$ and $9.5 \times 10^{-5}$ units, the value of the frequency being unchanged.

7 Let $z = f(x, y)$. Find $z$, if

(a) $dz = x^2 dx + y^2 dy$
(b) $dz = 2xy dx + x^2 dy$
(c) $dz = e^y dx + xe^y dy$

1.4 MULTIPLE INTEGRATION

1.4.1 Single Integration

The indefinite integral or antiderivative of a function $f(x)$ is a function $F(x)$ such that

$$\frac{d}{dx} F(x) = f(x).$$

The antiderivative $F$ is usually denoted by

$$\int f(x) \, dx + c.$$ 

Hence, a consequence of the definition is that

$$\frac{d}{dx} \left( \int f(x) \, dx \right) = f(x).$$

A Table of Integrals appears in Chapter 12.

If $F$ is the antiderivative of $f$ then the definite integral of a function $f$ is given by

$$\int_a^b f(x) \, dx = \left[ F(x) \right]_a^b = F(b) - F(a). \quad (i)$$

$a$ and $b$ are the limits of integration, and $x$ is called the dummy variable of integration.

The value of the integral depends on the function to be integrated, rather than the dummy used. Accordingly, we see that

$$\int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(u) \, du = F(b) - F(a).$$

The definite integral may be defined more formally as follows:
For each \( n = 2, 3, \ldots \) choose points \( x_0, x_1, \ldots, x_n \) such that \( a = x_0 < x_1 < \ldots < x_n = b \).
Form the sum
\[
\sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} f(x_i)\Delta x_i
\]
where \( \Delta x_i = x_{i+1} - x_i \) (see diagram) and let \( n \) tend to infinity. If the limit exists and doesn’t depend on the way the points \( x_i \) are chosen then define
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i)\Delta x_i.
\]

In fact, to avoid the problems that may occur if all of the points \( x_i \) "bunch" up at one end of the interval, we must insist that the maximum size for \( \Delta x_i \) converges to 0 instead of just asking that the number of points tends to infinity.

This result is usually given as the definition of the definite integral, and then equations (i) is derived from this definition, and is referred to as the **Fundamental Theorem of Integral Calculus**.

Finding a definite integral relates directly to finding the **area under the curve** represented by the function \( f(x) \) between the extremities \( a \) and \( b \).

### 1.4.2 Double Integration

Double integrals are useful in many applications. For instance, they can be used to obtain the centre of gravity of regions, moments of inertia, surface area and is used in various areas of potential theory. The following gives a method of evaluating double integrals.

The region \( R \) is **vertically simple** if it is described by the following
\[
a \leq x \leq b \quad \text{and} \quad g_1(x) \leq y \leq g_2(x)
\]
where \( g_1 \) and \( g_2 \) are continuous on \([a, b]\). This region appears in the figure to the right.

The region \( R \) is **horizontally simple** if it is described by the following
\[
c \leq y \leq d \quad \text{and} \quad h_1(y) \leq x \leq h_2(y)
\]
where \( h_1 \) and \( h_2 \) are continuous on \([a, b]\). This region appears in the figure to the right.

Let \( f(x, y) \) be a continuous function on the region \( R \). If \( R \) is the vertical simple region as given above then
\[
\int \int_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
\]
If $\mathbb{R}$ is the horizontally simple region as given above then
\[
\int \int_{\mathbb{R}} f(x, y) \, dA = \int_{c}^{d} \int_{h_{1}(y)}^{h_{2}(y)} f(x, y) \, dx \, dy.
\]

Note: Always sketch the region $\mathbb{R}$ of integration before attempting to evaluate double integrals.

Example

Evaluate $\int_{0}^{1} \int_{x^3}^{x^2} xy^2 \, dy \, dx$.

Method

The region $\mathbb{R}$ of integration for this integral is

$0 \leq x \leq 1$ and $x^3 \leq y \leq x^2$.

This region is the shaded region in the figure to the right.

The shaded region is vertically simple. The limits on the $y$ variable are going from the lower curve to the upper curve. The $x$ variable limits are from it’s minimum value to it’s maximum value.

When evaluating double integrals, we simply integrate the inner most integral first. This is then followed by the integration of the outer integral. In our example, we first integrate with respect to $y$ (treating the $x$ as a constant). Once the resulting function of $y$ is evaluated at the limits points, we then integrate with respect to $x$. (Note: the limits on the inner integral are functions of $x$ here). Thus,

\[
\int_{0}^{1} \int_{x^3}^{x^2} xy^2 \, dy \, dx = \int_{0}^{x^3} x \left[ \frac{y^3}{3} \right]_{x^3}^{x^2} \, dx \\
= \frac{1}{3} \int_{0}^{1} \left( x^7 - x^{10} \right) \, dx \\
= \frac{1}{88}.
\]

1.4.2 Reversing the Order of Integration

Example

(a) Refer to the previous example.

Instead of evaluating the integral $\int_{0}^{1} \int_{x^3}^{x^2} xy^2 \, dy \, dx$ with respect to $y$ first then $x$ we can reverse the order of integration. That is, we can integrate with respect to $x$ then $y$. However, we cannot simply interchange the limits around. Care must be taken when reversing the order of integration. We must look at the region $\mathbb{R}$ of integration that has been drawn and deduce the limits for the new integral

\[
\int \int_{\mathbb{R}} xy^2 \, dx \, dy.
\]

That is, we wish to change from a vertically (or $y$) simple region to a horizontally (or $x$) simple region of integration.
The questions we should ask ourselves when determining the limits on the integral are
(a) What are the lower and upper curves for the \( x \) variable?
(b) What are the intersection points for these lower and upper curves?
(c) What are the minimum and maximum values for \( y \)?

As a result we find that

\[
0 \leq y \leq 1 \quad \text{and} \quad y^{\frac{1}{3}} \leq x \leq y^{\frac{1}{2}}.
\]

Hence,

\[
\int \int_R xy^2 \, dx \, dy = \int_0^1 \int_{y^{1/3}}^{y^{1/2}} xy^2 \, dx \, dy
= \int_0^1 y^2 \left[ \frac{x^2}{2} \right]_{y^{1/3}}^{y^{1/2}} \, dy
= \frac{1}{2} \int_0^1 \left( y^{\frac{5}{3}} - y^3 \right) \, dy
= \frac{1}{88}.
\]

This is the same result as obtained by the previous double integral.

(b) By reversing the order of integration, evaluate

\[
\int_0^2 \int_{x^2}^4 xe^{-y^2} \, dy \, dx.
\]

**Method**

This integral cannot be evaluated as it stands. This is due to the fact that we cannot integrate the inner integral. That is

\[
\int_{x^2}^4 e^{-y^2} \, dy
\]

cannot be evaluated. However, if we change the order of integration we will see that we can evaluate the double integral. To do this we need to draw the region of integration.

From the limits of the integral we have that

\[
y \text{ goes from } y = x^2 \text{ to } y = 4
\]

and

\[
x \text{ goes from } x = 0 \text{ to } x = 2.
\]
Therefore, the region of integration is
\[(2,4)\]
\[y = x^2\]
\[R\]
(y-simple or vertically simple region.)

Upon reversing the order of integration we find that

\[x \text{ goes from } x = 0 \text{ to } x = \sqrt{y}\]

and

\[y \text{ goes from } y = 0 \text{ to } y = 4.\]

Hence,

\[
\int_0^2 \int_0^4 xe^{-y^2} \, dy \, dx = \int_0^4 \int_0^\sqrt{y} xe^{-y^2} \, dx \, dy
\]
\[
= \int_0^4 e^{-y^2} \left( \int_0^{\sqrt{y}} x \, dx \right) \, dy
\]
\[
= \int_0^4 e^{-y^2} \frac{x^2}{2} \bigg|_0^{\sqrt{y}} \, dy
\]
\[
= \frac{1}{2} \int_0^4 ye^{-y^2} \, dy
\]
\[
= -\frac{1}{4} e^{-y^2} \bigg|_0^4
\]
\[
= \frac{1}{4} (1 - e^{-16})
\]

1.4.4 A Geometrical Interpretation of \[\int \int_R f(x, y) \, dA\]

An element of area in the \(xy\) plane is given by

\[\Delta \text{Area} = \Delta x_i \Delta y_i.\]

That is, an element of volume enclosed by the surface

and the \(xy\) plane is given by

\[\Delta V_i = z_i \Delta x_i \Delta y_i\]

where \(\Delta x_i \) and \(\Delta y_i \) belong to the region \(R\) in the \(xy\) plane.
That is, if all the elementary volumes were added up over the region $R$ of the $xy$ plane we will obtain the volume enclosed by the surface above the region $R$ ( $\{[a,b] \times [c,d]\}$ - the rectangle) enclosed by the surface $z = f(x, y)$.

Therefore, volume below the surface $z = f(x, y)$ enclosed by the region $R$ can then be defined as

$$\int \int_{R} f(x, y) \, dA$$

where $\Delta x_i$ and $\Delta y_i$ are regarded as small enough to be approximated by $dx$ and $dy$. Also,

$$dA = dx \, dy \quad \text{or} \quad dA = dy \, dx.$$ 

**Note**: Volume is always a positive quantity.

**Example**

Find the volume of the solid that lies under the surface $z = 2x + 3y$ and above the region in the $xy$ plane that is bounded by the the curves $y = x^2$ and $y = x^3$.

**Method**

To find the volume of the solid it is recommended to at least draw the region $R$ of integration. This means we need to find any intersection points of the two curves, $y = x^2$ and $y = x^3$. That is, the points of intersection of the two curves is obtained finding the pointys that satisfy

$$y = x^2 \quad \text{and} \quad y = x^3.$$

Hence,

$$x^2 = x^3 \quad \implies \quad x^2(1 - x) = 0.$$ 

Therefore, the points of intersection of the two curves is $(0, 0)$ and $(1, 1)$.

Drawing the region of integration gives an indication of which variable to integrate with respect to first. This will depend on the $f(x, y)$ being integrable for all variables.

Therefore, the region of integration in this example is: From this figure we can choose either $x$ or $y$ to integrate with respect to first. We shall choose the $y$ variable. Therefore,

$y$ goes from $y = x^3$ to $y = x^2$ (ie. curve to curve) and

$x$ goes from $x = 0$ to $x = 1$ (ie minimum value to maximum value).
Hence, the volume of the solid is

\[
\int \int_R f(x, y) \, dA = \int_0^1 \int_{x^2}^{x^3} (2x + 3y) \, dy \, dx \\
= \int_0^1 2x \, y + \frac{3y^2}{2} \bigg|_{x^2}^{x^3} \, dx \\
= \int_0^1 2x(x^2 - x^3) + \frac{3}{2}(x^4 - x^6) \, dx \\
= \int_0^1 2x^3 - \frac{x^4}{2} - \frac{3x^6}{2} \, dx \\
= \left[ \frac{x^4}{2} - \frac{x^5}{10} - \frac{3x^7}{14} \right]_0^1 \\
= \frac{13}{70} \text{ cube units.}
\]

Note: Special Case

If \( f(x, y) = 1 \) then

\[
\int \int_R 1 \, dA = \text{Area bounded by a given curve in } R
\]

where \( R \) refers to the region in the \( xy \) plane.

Example

Find the area bounded by the lines \( y = x, \ x = 1 \) and the \( x \) axis in the first quadrant.

Method

We evaluating it is always good practice to draw a graph of the region \( R \) of integration. The region of integration is

\[ x \text{-simple or horizontally simple.} \]

We have a choice of which variable that we choose to integrate first. Usually this will depend on the region of integration. Here we will choose \( x \) first. Therefore,

\[ x \text{ goes from } x = y \text{ to } x = 1 \]

(from curve to curve) and

\[ y \text{ goes from } y = 0 \text{ to } y = 1 \]

(from minimum value to maximum value to completely define the region of integration).
Hence,

\[ \int \int_R dA = \int_0^1 \int_y^1 dx \, dy = \int_0^1 \left[ x \right]_y^1 dy = \int_0^1 (1 - y) dy = y - \frac{y^2}{2} \bigg|_0^1 = \frac{1}{2} \text{ square units}. \]

**Exercise 1D**

1. Evaluate the following integrals
   
   (a) \( \int_0^1 x e^x \, dx \) \hspace{1cm} (b) \( \int_0^\infty e^{-xu} \sin u \, du \)
   
   (c) \( \int \frac{1}{\sqrt{1-t^2}} \, dt \) \hspace{1cm} (d) \( \int_0^2 \frac{v}{\sqrt{1+v^2}} \, dv \)
   
   (e) \( \int \sec^2 x \, dx \) \hspace{1cm} (f) \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x| \, dx \)
   
   (g) \( \int \cos^{-1} u \, du \) \hspace{1cm} (h) \( \int_0^4 u e^{t-u} \, du \)
   
   (i) \( \int_0^1 (t-u) \sin u \, du \).

2. Evaluate the iterated integral, sketching the region of integration.
   
   (a) \( \int_0^1 \int_0^x xy \, dy \, dx \)
   
   (b) \( \int_0^1 \int_0^{\sqrt{x}} (2x-y) \, dy \, dx \)
   
   (c) \( \int_0^y \int_{-y}^{y+2} (x+2y^2) \, dx \, dy \).

3. Reverse the order of integration and then evaluate the resulting integral.
   
   (a) \( \int_0^1 \int_y^1 e^{-x^2} \, dx \, dy \)

4. Use double integration to find the area of the region in the \( xy \)-plane that is bounded by the given curves.
   
   (a) \( y = x, \ y^2 = x \).
   
   (b) \( x = 0, \ x = 1, \ y = 0 \text{ and } y = 2 \).

5. Using double integration, find the volume enclosed by the surface \( z = f(x,y) = xy \) and each of the regions as described in Q4.

6. Let \( R \) be the region bounded by the curves \( y = \sin x \) and \( y = \cos x \) for \( x \) lying between \( x = 0 \) and \( x = \frac{\pi}{4} \). Evaluate
   
   \( \int \int_R yx \, dA. \)

7. Let \( R \) be the region enclosed by the curves \( x = 0 \) and \( x = \sin y \) when \( y \) lies between \( y = 0 \) and \( y = \frac{\pi}{2} \). Evaluate
   
   \( \int \int_R x \cos y \, dA. \)
Chapter 2: First Order Differential Equations

2.1 DEFINITIONS

An ordinary differential equation is a relationship between functions of an independent variable, a dependent variable, and the ordinary derivatives of the dependent variable.

A partial differential equation is a relationship between functions of one or more independent variables, a dependent variable, and the partial derivatives of the dependent variable.

The order of an ordinary differential equation is the order of the highest derivative in the equation.

The degree of an ordinary differential equation is the power to which the highest order derivative in the equation is raised.

A linear differential equation of order \( n \) (with \( y \) dependent, \( x \) independent), is a differential equation of the form

\[
a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y = f(x),
\]

where \( a_0(x), a_1(x), \ldots, a_n(x) \), and \( f(x) \) are functions of \( x \) alone, and \( a_0(x) \) is not identically zero.

We say that the differential equation is linear in \( y \).

If each of the functions, \( a_i(x), i = 0, 1, 2, \ldots, n \), in this definition are constant functions, then the linear differential equation is said to be a linear differential equation of order \( n \) with constant coefficients.

A solution of a differential equation is any function (defined either implicitly or explicitly) which is free of derivatives and which satisfies identically the differential equation.

The general solution of an \( n \)th order ordinary differential equation is that solution which contains \( n \) arbitrary constants.

An initial value problem is a differential equation with certain imposed initial conditions which will allow for the determination of the arbitrary constants in the general solution, to form the particular solution.

Exercise 2A

1. Determine the order and degree of the following differential equations. List the dependent and the independent variables. State whether the equation is linear, linear with constant coefficients, or non-linear.

   (a) \( y + \frac{dy}{dx} = 0 \)

   (b) \( x^2 \left( \frac{d^2 y}{dx^2} \right)^2 + \left( \frac{dy}{dx} \right)^3 + y = 0 \)

   (c) \( \frac{d^2 t}{dx^2} + 5 \frac{dt}{dx} + 6t = e^x \)

   (d) \( \cos x \frac{dy}{dx} = x \)

   (e) \( y \frac{dy}{dx} = x \)

   (f) \( \frac{d^2 z}{dt^2} + \frac{dz}{dt} = t^4 \)

continued next page...
2 Write down an example of:
(a) a 2nd order linear differential equation with non-constant coefficients.
(b) a 2nd order, 3rd degree differential equation.
(c) a 1st order linear differential equation.

2.2 VARIABLE SEPARABLE EQUATION

A differential equation which can be written in the form
\[
\frac{dy}{dx} = f(x)g(y)
\]
is said to be a variables separable equation.

We solve a variables separable equation by "separating" the variables and integrating.

\[
\int \frac{dy}{g(y)} = \int f(x)dx + c.
\]

Since we have one arbitrary constant in the solution, we have found the general solution of the variables separable equation.

Exercise 2B

Solve the following differential equations.

(a) \[\frac{dy}{dx} = \frac{y(1 - y^2)}{x(1 - x^2)}\]
(b) \[\frac{dy}{dx} = \frac{x(1 - y^2)}{y(1 - x^2)}\]
(c) \[(x - 1)\frac{dy}{dx} = x(y + 1)\]
(d) \[y\ln xdx + (1 + 2y)dy = 0\]
(e) \[e^{x+y}\frac{dy}{dx} = e^{2x-y}.
\]

2.3 HOMOGENEOUS EQUATIONS

2.3.1 General Method

A first order differential equation \(\frac{dy}{dx} = f(x, y)\) which can be written in the form
\[
\frac{dy}{dx} = F\left(\frac{y}{x}\right),
\]
is called a homogeneous differential equation.

We solve the homogeneous equation by letting \(v = \frac{y}{x}\) That is let \(y = xv\).
Using the "product rule"
\[
\frac{dy}{dx} = x \frac{dv}{dx} + v ,
\]
the equation then becomes
\[
x \frac{dv}{dx} + v = F(v) .
\]
Hence
\[
x \frac{dv}{dx} = F(v) - v .
\]
This equation is clearly separable, and can be solved as such.

### 2.3.2 Homogeneous Equations Requiring a Change of Variables (LINEAR SHIFT)

The differential equation
\[
\frac{dy}{dx} = ax + by + c \frac{dx}{dx} + ey + f
\]
can usually be converted to a homogeneous differential equation by a change of variable of the form
\[
x = X + p , \quad y = Y + q
\]
where \( p \) and \( q \) are chosen appropriately (as the point of intersection of the straight lines). The reduced equation becomes
\[
\frac{dY}{dX} = \frac{aX + bY}{dX + cY}.
\]

**Example**

Solve the differential equation
\[
\frac{dy}{dx} = 2x - y + 3 \frac{x + 2y - 1}{x + 2y - 1}.
\]

**Method**

Let \( x = X + p \) and \( y = Y + q \) then \( \frac{dy}{dx} = \frac{dY}{dX} \). Upon substituting into the given differential equation we have
\[
\frac{dY}{dX} = \frac{2(X + p) - (Y + q) + 3}{(X + p) + 2(Y + q) - 1} = \frac{2X - Y + (2p - q + 3)}{X + 2Y + (p + 2q - 1)}.
\]

To find the values of \( p \) and \( q \) we let
\[
2p - q + 3 = 0
\]
and
\[
p + 2q - 1 = 0.
\]

Upon solving these two equations we find that \( p = -1 \) and \( q = 1 \) and the differential equation becomes
\[
\frac{dY}{dX} = \frac{2X - Y}{X + 2Y}
\]
which is a homogeneous differential equation. To solve this equation we let \( V = Y/X \) then
\[
\frac{dy}{dx} = X \frac{dV}{dx} + V.
\]
Therefore, the differential equation becomes

\[ X \frac{dV}{dX} + V = \frac{2 - V}{1 + 2V}. \]

That is,

\[ \frac{dX}{X} = \frac{2V + 1}{-2(V^2 + V + 1)} \, dV \]

upon separating the variables. Hence,

\[ V^2 + V + 1 = \ln \left( \frac{c}{X^2} \right). \]

Replacing \( V \) by \( \frac{Y}{X} \) we have that

\[ \frac{Y^2}{X^2} + \frac{Y}{X} + 1 = \ln \left( \frac{c}{X^2} \right) \]

and hence the solution to the given differential equation can written in the form

\[ \frac{(y - 1)^2}{(x + 1)^2} + \frac{y - 1}{x + 1} + 1 = \ln \left( \frac{c}{(x + 1)^2} \right). \]

**Exercise 2C**

Solve the following differential equations.

1. \( x \frac{dy}{dx} = y + \sqrt{x^2 + y^2} \)
2. \( \frac{dy}{dx} = \frac{4y^2}{x^2} + \frac{5y}{x} + 1 \)
3. \( (x^2 - y^2) \, dx + 2xy \, dy = 0 \)
4. \( \frac{dy}{dx} + \frac{x + y}{x} = 0 \)
5. \( \frac{dy}{dx} = \frac{y - x + 1}{y + x + 5} \)
6. \( \frac{dy}{dx} + \frac{2x - y - 4}{2y - x + 5} = 0 \)
7. \( \frac{dy}{dx} = \frac{x - y + 2}{x - y + 1} \)
8. \( \frac{dy}{dx} + \frac{2x - y - 4}{2x - y + 5} = 0 \)
9. \( (x - 2y + 1) \, dx + (4x - 3y - 6) \, dy = 0 \)
10. \( (5x + 2y + 1) \, dx + (2x + y + 1) \, dy = 0 \)
11. \( (3x - y + 1) \, dx - (6x - 2y - 3) \, dy = 0 \)
12. \( \frac{dy}{dx} = -\frac{2x + 3y + 1}{4x + 6y + 1} \)
2.4 THE LINEAR FIRST ORDER EQUATION

A linear first order differential equation has the form

\[ \frac{dy}{dx} + p(x)y = q(x). \]

Firstly, check to determine whether or not the left hand side of the equation is already the exact derivative of a product. If it is, then integrate directly. Otherwise, ensure that the co-efficient of \( \frac{dy}{dx} \) is 1.

We then solve this equation by multiplying both sides of the equation by an appropriate function (called an integrating factor), chosen in such a way that the LHS of the differential equation will now be exactly the derivative of a product.

It is then possible to directly integrate both sides of the "new" equation.

We wish to find \( R(x) \) so that

\[ R(x)\frac{dy}{dx} + R(x)p(x)y = R(x)q(x) \]

is directly integrable. That is, the left hand side of (*) becomes

\[ \frac{d}{dx}(R(x)y). \]

This is so provided the LHS is exactly the derivative of the product. That is,

\[ \frac{d}{dx}(R(x)y) = R(x)\frac{dy}{dx} + y\frac{dR(x)}{dx} \]

By equating the co-efficients of \( y \) with (*), we obtain

\[ \frac{dR(x)}{dx} = R(x)p(x). \]

This equation is a variables separable equation. Thus,

\[ \frac{dR}{R} = p(x)dx \]

\[ \ln R = \int p(x)dx \]

Hence,

\[ R(x) = e^{\int p(x)dx}. \]

and (*) becomes

\[ \frac{d}{dx}(R(x)y) = R(x)q(x). \]

Substituting and integrating, we obtain

\[ R(x)y = \int R(x)q(x)dx + c. \]
Example

Solve the following linear differential equation subject to the given initial condition.

\[ x \frac{dy}{dx} + y = xe^x, \quad y(1) = 0. \]

Method

Rewriting the given differential equation in linear form gives

\[ \frac{dy}{dx} + \frac{1}{x}y = e^x \]

where

\[ R(x) = e^{\int \frac{1}{x} \, dx} \quad \text{and} \quad q(x) = e^x. \]

Thus,

\[ R(x) = x. \]

Multiplying both sides of the linear form of the differential equation we find that

\[ \frac{d}{dx} (R(x)y) = R(x)q(x). \]

That is,

\[ \frac{d}{dx} (xy) = xe^x. \]

Upon integration we have

\[ xy = \int xe^x \, dx + c \]
\[ = xe^x - \int e^x \, dx + c \]
\[ = xe^x - e^x + c. \]

Hence, the solution to the given differential equation is

\[ y = e^x \left( 1 - \frac{1}{x} \right) + \frac{c}{x}. \]

using the initial condition we can find the arbitrary constant. That is,

\[ y(1) = 0 \quad \implies \quad c = 0. \]

Therefore,

\[ y = e^x \left( 1 - \frac{1}{x} \right) \quad \text{for} \quad x \neq 0. \]

The graph to the right shows the solution curve.
Exercise 2D

1. Solve the following differential equations.
   (a) \( \frac{dy}{dx} - y \tan x = 2 \sin x \)
   (b) \( \frac{dy}{dx} + y \cot x = \cos 3x \)
   (c) \( (1 - x^2) \frac{dy}{dx} - xy = 1 \)
   (d) \( 2(1 - x^2) \frac{dy}{dx} - (1 + x)y = (1 - x^2)^{\frac{3}{2}} \)

2. Solve the following differential equations subject to the given initial condition.
   (a) \( \frac{dy}{dx} + \frac{1}{x} y = x, \quad y(1) = 0 \)
   (b) \( \frac{dy}{dx} = y + e^x, \quad x = 0, y = 1 \)
   (c) \( \frac{dy}{dx} = xe^{-x^2} - 2xy, \quad x = 0, y = 2 \)

3. A body of mass \( m \) falls where the air resistance to the motion is proportional to the speed of the body. The equation of motion is

\[
 m \frac{dv}{dt} = mg - kv
\]

where \( g \) is the acceleration due to gravity, \( v \) is the velocity of the body at time \( t \) and \( k \) is a constant. Find the velocity \( v(t) \) if the body is released from rest.

2.5 BERNOULLI’S EQUATION

A first order differential equation which can be written in the form

\[
y' + p(x)y = q(x)y^n
\]

is called a Bernoulli equation.

If we substitute \( v = y^{1-n} \) we have

\[
 \frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}.
\]

Hence,

\[
 \frac{dy}{dx} = \frac{1}{1-n} y^n \frac{dv}{dx}.
\]

Substituting into the differential equation, we obtain

\[
 \frac{1}{1-n} y^n \frac{dv}{dx} + p(x)y = q(x)y^n
\]

and hence

\[
 \frac{1}{1-n} \frac{dv}{dx} + p(x)y^{1-n} = q(x).
\]

However \( v = y^{1-n} \). Thus, on simplifying we obtain

\[
 \frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x).
\]

This equation is linear in \( x \) and \( v \) and can be solved as such.
Exercise 2E  Miscellaneous

Solve the following differential equations or initial value problems.

(a) \( xy' - 3y = x^3 \), \( y(1) = 0 \)

(b) \( x(x + 1)y' - y = 2x^2(x + 1) \)

(c) \( xy' + y + x^2 y^2 e^x = 0 \)

(d) \( xy' = y - x e^y \)

2.6 FIRST ORDER EXACT EQUATION

Recall that the total differential of the function \( f(x, y) = c \) is given by

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy = 0
\]

which can be rewritten as the differential equation

\[ M \, dx + N \, dy. \]

Hence, if we are given the first order differential equation of the form

\[ M \, dx + N \, dy = 0 \]

then provided \( M \) and \( N \) are functions of the form

\[
M = \frac{\partial f}{\partial x} \quad \text{and} \quad N = \frac{\partial f}{\partial y}
\]

with respect to a function \( f(x, y) \), then the differential equation becomes

\[
\frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy = 0.
\]

That is,

\[ df = 0. \]

A general solution of this equation is

\[ f(x, y) = c \]

where \( c \) is an arbitrary constant.

Condition for Exactness

The condition that must be satisfied for this to occur is

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.
\]

If this is so, then we say the differential equation is exact.
**Example**

Determine if the following differential equation is exact.

\[ \frac{dy}{dx} = -\frac{2xy^3}{1 + 3x^2y^2}. \]

**Method**

Rewriting the given differential equation, we have that

\[ 2xy^3 \, dx + (1 + 3x^2y^2) \, dy = 0. \]

Hence,

\[ M = 2xy^3 \quad \text{and} \quad N = 1 + 3x^2y^2. \]

Now

\[ \frac{\partial M}{\partial y} = 6xy^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy^2. \]

Hence,

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \]

Therefore, the differential equation is exact.

**Method of Solution for an Exact Equation**

Consider the differential equation

\[ M \, dx + N \, dy. \]

1. Check that the given differential equation is exact. That is,

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \]

2. Integrate \( M(x, y) \) with respect to \( x \) adding an arbitrary constant of \( y \), say \( g(y) \).

3. Differentiate the result in 2. with respect to \( y \) and set result equal to \( N(x, y) \). Hence, solve for \( g'(y) \).

4. Find an expression for \( g(y) \) and substitute into the result for \( f(x, y) \) found in 2.

5. Consequently, a general solution is found to be

\[ f(x, y) = c \]

where \( c \) is a constant.
Example

Consider the preceding example where

\[ M = \frac{\partial f}{\partial x} = 2xy^3 \quad \text{and} \quad N = \frac{\partial f}{\partial y} = 1 + 3x^2y^2. \]

Method

Now

\[ M = \frac{\partial f}{\partial x} = 2xy^3 \quad \Rightarrow \quad f(x, y) = \int 2xy^3 \, dx + g(y) \]

\[ = x^2y^3 + g(y). \]

Now differentiating \( f(x, y) \) partially with respect to \( y \), we obtain that

\[ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2y^3 + g(y)) \]

\[ = 3x^2y^2 + g'(y) \]

\[ = N. \]

That is,

\[ N = 1 + 3x^2y^2 = 3x^2y^2 + g'(y). \]

Therefore,

\[ g'(y) = 1. \]

Upon integrating we find that \( g(y) = y \). Here the constant of integration becomes superfluous to the solution of the given differential equation because \( f(x, y) = c \) where \( c \) is a constant is the solution to the differential equation.

Thus,

\[ f(x, y) = 3x^2y^2 + y \]

Hence,

\[ 3x^2y^2 + y = c \]

is the solution to the given differential equation.

Solution Curves

Below is a graph of a selection of the family of solution curves for the above differential equation.
The specific solution curve will be determined by the given initial condition.

**Exercise 2F**

1. Determine which of the following equations is exact. If it is exact, solve it.
   
   (a) \((2x + 3) + (2y - 2)y' = 0\)
   
   (b) \((2x + 4y) + (2x - 2y)y' = 0\)
   
   (c) \(\frac{xdx}{(x^2 + y^2)^{3/2}} + \frac{ydy}{(x^2 + y^2)^{3/2}} = 0\)
   
   (d) \(\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0\)

2. Find the values of \(b\) for which the following equations are exact, and solve them for that value of \(b\).
   
   (a) \((xy^2 + bx^2y)dx + (x + y)x^2dy = 0\)
   
   (b) \((y^e^{xy} + x)dx + bx e^{xy}dy = 0\)

3. Solve the following differential equations.
   
   (a) \((3y^2 - x)\frac{dy}{dx} = y\)
   
   (b) \(\frac{dy}{dx} = \frac{y(e^{xy} - 2x)}{x(x - e^{xy})}\)
   
   (c) \((ye^x - 2x)dx + e^x dy = 0\)
   
   (d) \(\cos x \sec y dx + \sin x \tan y \sec y dy = 0\)

### 2.7 INTEGRATING FACTORS AND EXACT EQUATIONS

Consider the first order differential equation of the form

\[ Mdx + Ndy = 0. \]

If this equation is not already exact, we can often make it so by multiplying throughout by an integrating factor \(R(x, y)\) and then finding \(\frac{\partial M}{\partial y}\) and \(\frac{\partial N}{\partial x}\). The integrating factors are able to be determined by solving

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

for \(R\). It will usually be in one of the following forms.

(a) \(R = R(x)\)

(b) \(R = R(y)\)

(c) \(R = R(x, y) = x^m y^n\)

Otherwise, further information will need to be given.

*Example*

Solve the differential equation

\[(2y^2 + 4x^2 y)dx + (4xy + 3x^3)dy = 0.\]

*Method*

We see that

\[ M = 2y^2 + 4x^2 y \quad \text{and} \quad N = 4xy + 3x^3. \]
Thus, \[
\frac{\partial M}{\partial y} = 4y + 4x^2 \text{ and } \frac{\partial N}{\partial x} = 4y + 9x^2.
\]
Since, \[
\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},
\]
then the equation is not exact.

(a) Let \( R = R(x) \). After multiplying the given differential equation throughout by \( R \) we require for exactness that
\[
\frac{\partial (R(x)M)}{\partial y} = \frac{\partial (R(x)N)}{\partial x}.
\]
That is,
\[
R(x)\frac{\partial M}{\partial y} = R(x)\frac{\partial N}{\partial x} + NR'(x).
\]
Upon substituting for \( M \) and \( N \) and simplifying we have that
\[
\frac{R'(x)}{R(x)} = \frac{-5x^2}{4xy + 3x^3}.
\]
However, this is not possible as \( R \) is a function of \( x \) only. Therefore, the integrating factor is not a function of \( x \).

(b) Let \( R = R(y) \). After multiplying the given differential equation throughout by \( R \) we require for exactness that
\[
\frac{\partial (R(y)M)}{\partial y} = \frac{\partial (R(y)N)}{\partial x}.
\]
That is,
\[
R(y)\frac{\partial M}{\partial y} + MR'(y) = R(y)\frac{\partial N}{\partial x}.
\]
Upon substituting for \( M \) and \( N \) and simplifying we have that
\[
\frac{R'(y)}{R(y)} = \frac{5x^2}{2y^2 + 4x^2}.
\]
However, this is not possible as \( R \) is a function of \( y \) only. Therefore, the integrating factor is not a function of \( y \).

(c) Let \( R = x^m y^n \). After multiplying the given differential equation throughout by \( R \) we require for exactness that
\[
\frac{\partial (RM)}{\partial y} = \frac{\partial (RN)}{\partial x}.
\]
That is,
\[
R\frac{\partial M}{\partial y} + M\frac{\partial R}{\partial y} = R\frac{\partial N}{\partial x} + N\frac{\partial R}{\partial x}.
\]
Upon substituting for \( R, M \) and \( N \) and simplifying we have that
\[
x^{m+2}y^n(4n - 3m - 5) + x^m y^{n+1}(2n - 4m) = 0.
\]
Equating co-efficients we have that
\[
\begin{align*}
4n - 3m - 5 &= 0 \\
2n - 4m &= 0.
\end{align*}
\]
Therefore, 

\[ m = 1 \quad \text{and} \quad n = 2. \]

Hence, the integrating factor is \( R(x, y) = xy^2 \).

**To Find the General Solution**

Let

\[ \frac{\partial f(x, y)}{\partial x} = R(x, y)M \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = R(x, y)N. \]

Upon integration we have that

\[ f(x, y) = \int R(x, y)M \, dx \]
\[ = \int (2xy^4 + 4x^3y^3) \, dx + g(y). \]
\[ = x^2y^4 + x^4y^3 + g(y) \quad (\ast) \]

Now differentiating \((\ast)\) with respect to \( y \) we have that

\[ \frac{\partial f(x, y)}{\partial y} = 4x^2y^3 + 3x^4y^2 + g'(y) \]
\[ = R(x, y)N \]
\[ = xy^2(4xy + 3x^3). \]

That is, \( g'(y) = 0 \). Therefore, \( g(y) = c^* \), where \( c^* \) is a constant. Hence,

\[ f(x, y) = x^2y^4 + x^4y^3 + c^*. \]

Note that the solution to the given differential equation should be of the form \( f(x, y) = k \). Therefore, the solution to the given differential equation is

\[ x^2y^4 + x^4y^3 = k. \]

**Exercise 2G**

1 Solve the differential equation

\[ (3xy + y^2) + (x^2 + xy)y' = 0 \]

using the integrating factor

\[ R(x, y) = \frac{1}{xy(2x + y)}. \]

2 Find the integrating factors to make the following differential equations exact, and then solve them.

(a) \( (3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0 \)

(b) \( dx + \left( \frac{x}{y} - \sin y \right) \, dy = 0 \)

(c) \( (3x + \frac{6}{y}) \, dx + \left( \frac{x^2}{y} + \frac{3y}{x} \right) \, dy = 0. \)

continued next page...
3. Show that if

\[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = Q \]

where \( Q = Q(y) \), then the differential equation

\[ M + Ny' = 0 \]

has an integrating factor of the form

\[ R(y) = e^{\int Q(t)\,dt}. \]

2.8 SUMMARY OF FIRST ORDER EQUATIONS

The first step in solving a differential equation is to classify the equation according to order and degree. This step allows us to review the techniques we have available for solving an equation of the given type, and to compare the number of constants in the general solution with the order of the equation.

1. If the given differential equation is a first order equation, then we ask the following questions in this specific order.

   Is the equation variables separable? If it is, we separate the variables and integrate.

2. Is the equation linear? If it is, we find an appropriate integrating factor and solve the equation.

3. Is the equation homogeneous? If it is, we let \( v = \frac{y}{x} \) then \( \frac{dy}{dx} = x \frac{dv}{dx} + v \) and separate variables in the resulting differential equation.

4. Is the equation a Bernoulli’s equation? If it is, reduce it to a linear equation, and use 2.

5. Is the equation exact? If it is, proceed to find the function whose total differential generates the differential equation.

   If not, can we find an integrating factor to make it exact?

6. If all the above fail, we can try interchanging the roles of the dependent and independent variables in 2 and see if the equation is linear in the "new" variables.

Exercise 2H (Revision of First Order Equations)

1. Solve the following differential equations.
   
   (a) \( (1 - x^2) \frac{dy}{dx} + xy = 3x \)
   
   (b) \( x \frac{dy}{dx} = y + 2\sqrt{y^2 - x^2} \)
   
   (c) \( y'(2x + y^2) = y \)
   
   (d) \( \frac{dy}{dx} - y \tan x = -y^2 \sec x \)
   
   (e) \( x^3 \frac{dy}{dx} + 3y^2 = xy^2 \)
   
   (f) \( (2x - ye^{xy})dx + (2y - xe^{xy})dy = 0 \)
   
   (g) \( (y^2 - 1) \frac{dy}{dx} + (x - 1)e^x y^2 = 0 \)
   
   (h) \( x(1 + x) \frac{dy}{dx} - x^2 = y(1 + x) \)
   
   (i) \( x \frac{dy}{dx} + 3y = \frac{1}{x} \)
   
   (j) \( \frac{dy}{dx} - y = xy^{1/2} \)
   
   (l) \( (2xy + e^y)dx + (x^2 + xe^y)dy = 0 \)
2 Solve the initial value problem

\[ y' + 2y = g(x), \quad y(0) = 0 \]

where

\[ g(x) = \begin{cases} 
1, & 0 \leq x \leq 1 \\
0, & x > 1 
\end{cases} \]

Hint: Solve the differential equation separately for \( 0 \leq x \leq 1 \) and for \( x > 1 \). Then match the solutions so that \( y \) is continuous at \( x = 1 \).

Note: It is impossible to make both \( y \) and \( y' \) continuous at \( x = 1 \).

3 Solve the following differential equations

(a) \[ \frac{dy}{dx} = \frac{2x + 3y + 4}{3x + y - 1} \]

(b) \( x(1 - x^2) \, dy = (x^2 - x + 1) \, dx \)

(c) \( y^2 \, dx + (y^2x + 2xy - 1) \, dy = 0 \)

(d) \( \frac{dy}{dx} + 3y = \frac{1}{x} \)

(e) \( \frac{dy}{dx} = \frac{x - y + 2}{x - y + 1} \).

4 (a) Find integrating factors to make the following differential equations exact, and then solve them.

(i) \( xy^4 \, dx + (x^2y^3 - y + 1) \, dy = 0 \)

(ii) \( (x^2y^2 + y) \, dx + (2x^3y - x) \, dy = 0 \)

(iii) \( x(x - y) \frac{dy}{dx} - y(2x - y) = 0 \)

(iv) \( 3y(ydx + 3xdy) = 2x^2(3ydx + 2xdy) \).

(b) Solve each of the equations in (a) by as many other techniques as possible.
Chapter 3: Second Order Differential Equations

3.1 REDUCTION OF ORDER

3.1.1 y-absent or x-absent Differential Equations

A second order differential equation with y-absent or x-absent (i.e. missing from the equation) will, after an appropriate substitution, reduce to a first order equation.

3.1.2 y-absent

A y-absent differential equation is a relation involving \( x \), \( \frac{dy}{dx} \), and \( \frac{d^2y}{dx^2} \), with no explicit function of \( y \) being present. For example,

\[ 3y''(x) - 2y'(x) = \sin x \]

is a \( y(x) \) absent second order linear differential equation.

It can be solved by letting \( p = \frac{dy}{dx} \), then \( \frac{dp}{dx} = \frac{d^2y}{dx^2} \), and the equation is now in terms of \( p \) and \( x \).

3.1.3 x-absent

An x-absent differential equation is a relation involving \( y \), \( \frac{dy}{dx} \), and \( \frac{d^2y}{dx^2} \), without an explicit function of \( x \) being present. For example,

\[ 3y''(x) - 2y(x)y'(x) + y(x) = 0 \]

is a \( x \) absent second order non-linear differential equation.

It can be solved by letting \( p = \frac{dy}{dx} \), then \( p = \frac{dy}{dx} = \frac{d^2y}{dx^2} \).

However, in this case, the equation still involves an \( x \) in the \( \frac{dp}{dx} \) term. Therefore, we must apply the chain rule to \( \frac{dp}{dx} \).

\[
\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dy}{dx} \frac{dp}{dy} = p \frac{dp}{dy}.
\]

Now, the equation will be in terms of \( y \) and \( p \).
3.1.4 General Principle of Reduction of Order

If we know that \( y_1 = f_1(x) \) is a solution to the given second order linear homogeneous differential equation, we try \( y = uy_1 \) as the general solution and attempt to find \( u \). This will change the original second order equation to a second order \( u \)-absent equation, which, by a further change of variable, can be reduced to a first order equation. Notice that \( y_1 = x \) is always a solution to an equation of the form

\[
f(x)y'' + g(x)[xy' - y] = 0.
\]

Exercise 3A

1. Solve the following differential equations.
   (a) \( \frac{d^2y}{dx^2} + n^2x = 0 \)
   (b) \( \cos^2x \frac{d^2y}{dx^2} = 1 \)
   (c) \( y \frac{d^2y}{dx^2} + 1 = \left( \frac{dy}{dx} \right)^2 \)
   (d) \( y'' - yy' = 0 \)
   (e) \( (1 + x^2)y'' = 1 \)
   (f) \( y'' + x(y')^2 = 0 \)
   (g) \( yy'' + (y')^2 = 0 \)
   (h) \( \frac{d^2y}{dx^2} + n^2y = 0 \)
   (i) \( xy''' + y'' = 1 \)

2. For each of the following differential equations, check that \( y_1 \) is a solution of the corresponding homogeneous equation and then calculate the general solution by the reduction of order method.
   (a) \( y'' + 8y' + 16y = 0; \quad y_1 = e^{-4x} \)
   (b) \( (1 + x^2)y'' + xy' - y = 0; \quad y_1 = x \)
   (c) \( y'' - 4y' - 5y = 0; \quad y_1 = e^{-x} \)
   (d) \( y'' + 4y = 0; \quad y_1 = \cos 2x \)
   (e) \( y'' - 4y' + 3y = 3x - 4; \quad y_1 = e^x \)

3.2 CONSTANT COEFFICIENTS LINEAR DIFFERENTIAL EQUATIONS

The second order linear non-homogeneous differential equation with constant coefficients has the form

\[
ay'' + by' + cy = f(x),
\]

where \( a, b, c \) are constants, and \( a \neq 0 \), with its corresponding homogeneous equation

\[
ay'' + by' + cy = 0.
\]

The general solution of the homogeneous equation is called the complementary function of the general solution of the non-homogeneous equation and is usually denoted by \( y_c \). It is found by finding the roots \( m_1 \) and \( m_2 \) of the auxiliary or characteristics equation

\[
am^2 + bm + c = 0.
\]
This is obtained by assuming that every solution of a constant co-efficient differential equation has a solution of the form \( y = e^{mx} \).

To determine the complementary function, we have 3 cases to consider:

(i) \( b^2 - 4ac > 0 \) : \( y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} \)

(ii) \( b^2 - 4ac = 0 \) : \( y_c = (c_1 x + c_2) e^{m_1 x} \)

(iii) \( b^2 - 4ac < 0 \) : \( y_c = e^{rx} [A \cos sx + B \sin sx] \)

where \( m_1 = r + is \), and \( m_2 = r - is \).

**Example**

Find the characteristic equation and hence, the general solution of

\[ y'' + 4y' + 20y = 0. \]

**Method**

Upon substituting \( y = e^{mx} \) into the given homogeneous differential equation we obtain

\[ m^2 e^{mx} + 4m e^{mx} + 20 e^{mx} = 0. \]

Dividing through by \( e^{mx} \), we obtain the characteristic equation

\[ m^2 + 4m + 20 = 0. \]

The roots of this equation are \( m = -2 \pm 4i \). Therefore, the general solution to the given differential equation is

\[ y(x) = e^{-2x} (c_1 \cos 4x + c_2 \sin 4x). \]

**General Solution to a Non-Homogeneous Differential Equation**

The general solution \( y \) of the equation

\[ ay'' + by' + cy = f(x) \]

is of the form

\[ y = y_c + y_p \]

where \( y_c \) is the complementary function and \( y_p \) is the particular integral. \( y_c \) is of the form

\[ y_c = c_1 y_1 + c_2 y_2 \]

where \( y_1 \) and \( y_2 \) each separately satisfy the homogeneous equation.

If the RHS of the non-homogeneous differential equation is the sum of two or more functions that is,

\[ \text{RHS} = f_1(x) + f_2(x) \]
then the particular integral will be the sum of the particular integrals $y_p^1$ and $y_p^2$ calculated separately for each of $f_1(x)$ and $f_2(x)$ respectively. That is,

$$y_p = y_p^1 + y_p^2$$

### 3.3 OPERATOR D

If we let

$$Dy = \frac{dy}{dx} \quad \text{and} \quad D^2y = \frac{d^2y}{dx^2}$$

then the equation

$$ay'' + by' + cy = f(x)$$

can be written in the form

$$aD^2y + bDy + cy = f(x)$$

or more simply as

$$(aD^2 + bD + c)y = f(x).$$

We write

$$\frac{1}{P(D)}f(x) = Q(x) \quad \text{to mean} \quad f(x) = P(D)Q(x).$$

Further, since $P(D)$ is a linear operator, $\frac{1}{P(D)}$, when it exists, is also a linear operator.

**Rules for finding $\frac{1}{P(D)}f(x)$, for selected functions $f$**

We wish to solve $P(D)y = f(x)$, where $P(D) = aD^2 + bD + c$.

**STEP 1:** Find $y_c$ by solving $P(D)y_c = 0$.

**STEP 2:** Find $y_p$ by solving $y_p = \frac{1}{P(D)}f(x)$.

**STEP 3:** The general solution is $y = y_c + y_p$.

Regarding STEP 2, we have the following table, because $y_p$ depends on the function $f$:
<table>
<thead>
<tr>
<th>TYPE $f(x)$</th>
<th>Method to find $y_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial of degree $n$</td>
<td>1 Use ‘long division’ or the Binomial Theorem. Keep only terms involving $1, D, D^2, \ldots, D^n$.</td>
</tr>
<tr>
<td>$e^{\alpha x}$</td>
<td>2 Use $\frac{1}{P(D)} e^{\alpha x} = \frac{e^{\alpha x}}{P(\alpha)}$, $P(\alpha) \neq 0$. $P(\alpha) = 0$ will occur when $c_1 e^{\alpha x}$ is included in $y_c$. In this case write $e^{\alpha x}$ as $e^{\alpha x} e^0$ and treat as Type 3. This technique will always work.</td>
</tr>
<tr>
<td>$e^{\alpha x} F(x)$</td>
<td>3 Use Operator Shift Theorem. $\frac{1}{P(D)} e^{\alpha x} F(x) = \frac{e^{\alpha x}}{P(D + \alpha)} F(x)$. $F(x)$ a polynomial. $F(x)$ an exponential function. $F(x)$ an trigonometric function. This technique never fails.</td>
</tr>
<tr>
<td>$\cos \alpha x$ or $\sin \alpha x$</td>
<td>4 Use $\frac{1}{P(D^2)} \cos \alpha x = \frac{1}{P(-\alpha^2)} \cos \alpha x$, provided $P(-\alpha^2) \neq 0$. $P(-\alpha^2) = 0$ will occur when $y_c$ contains the function $c_1 \cos \alpha x + c_2 \sin \alpha x$.</td>
</tr>
</tbody>
</table>
| $\cos \alpha x$ or $\sin \alpha x$ and $P(-\alpha^2) = 0$ | 5 In this case we write $\cos \alpha x$ as $Re\left(e^{i\alpha x}\right)$ or $\sin \alpha x$ as $Im\left(e^{i\alpha x}\right)$.

We then use either $\frac{1}{P(D)} Re\left(e^{i\alpha x}\right) = Re\left(\frac{1}{P(D)} e^{i\alpha x} e^{0x}\right)$ or $\frac{1}{P(D)} Im\left(e^{i\alpha x}\right) = Im\left(\frac{1}{P(D)} e^{i\alpha x} e^{0x}\right)$, and the Operator Shift Theorem. |
3.4 EULER’S EQUATION

Euler’s equation is of the form

\[ x^2 \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + by = f(x). \]

Using the substitution

\[ x = e^z \quad \text{or} \quad z = \ln x, \]

Euler’s equation may be reduced to

\[ \frac{d^2 y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + by = g(z), \]

which is an equation with constant coefficients.

Example

Solve the differential equation

\[ xy'' - 3y' + 3\frac{y}{x} = 0. \]

Method

After multiplying through by \( x \) the differential equation becomes

\[ x^2 y'' - 3xy' + 3y = 0 \]

which is in the form of Euler’s equation.

Let

\[ x = e^z \]

then

\[ dx = e^z dz \quad \Rightarrow \quad \frac{d}{dx} = e^{-z} \frac{d}{dz}. \]

Therefore,

\[
\frac{d^2}{dx^2} = \frac{d}{dx} \left( \frac{e^{-z} d}{dz} \right) \\
= e^{-z} \frac{d}{dz} \left( \frac{e^{-z} d}{dz} \right) \\
= e^{-2z} \frac{d^2}{dz^2} - e^{-2z} \frac{d}{dz}.
\]

Upon substituting for \( x, \frac{d}{dx} \) and \( \frac{d^2}{dx^2} \) into the given equation we have

\[
e^{2z} \left( e^{-2z} \frac{d^2 y}{dz^2} - e^{-2z} \frac{dy}{dz} \right) - 3e^z \left( e^{-z} \frac{dy}{dz} \right) + 3y = 0.
\]

That is,

\[
\frac{d^2 y}{dz^2} - 4 \frac{dy}{dz} + 3y = 0
\]

which is a constant co-efficient differential equation. Upon solving this differential equation we have

\[ y = Ae^z + Be^{3z} \]

\[ = Ax + B x^3. \]
A graph of a selection of solutions curves is found below:

The correct solution curve will depend on the given conditions.

**Exercise 3B**

Solve the following differential equations

1. \( y'' - 2y' - 8y = 4 \)
2. \( 2y'' + 5y' + 2y = x^2 \)
3. \( y'' + 8y' + 16y = 25e^x \)
4. \( y'' + 4y' = 8x + 6 \sin 2x \)
5. \( y'' + 6y' + 9y = 2e^{-x} \)
6. \( y'' + 4y = \cos 2x \)
7. \( 2y'' + y' - y = \cos x - 3 \sin x \)
8. \( y'' - y = \cosh x \)
9. \( y'' - 2y' + y = 6e^{-x} + 4e^{-3x} + xe^x \)
10. \( y'' + 4y' + 4y = 16xe^{-2x} \)
11. \( y'' + 2y = (3 + 4x)e^x \)
12. \( y'' + 4y' + 4y = 12x^2e^{-2x} \)
13. \( y'' - 4y' + 4y = 4e^{2x} \)
14. \( y'' - 3y' + 2y = e^x \)
15. \( (D^4 + 5D^2 + 4)y = \cos x \)
16. \( x^2y'' + 2xy' - 2y = x^3 \)
17. \( x^2y'' + 3xy' = 2x \)
18. \( x^2y'' - 3xy' + 3y = 0 \) given that \( y = 5 \) and \( y'' = 12 \) at \( x = 1 \)
19. \( x^2y'' - xy' + 2y = \ln x \)
Chapter 4: Special Functions

There are some special functions that we encounter in mathematics that are the essential building blocks that represent important solutions to various applied problems. Some of these special functions will be given in this chapter.

4.1 DERIVATIVE OF AN INTEGRAL

4.1.1 The Fundamental Theorem of Calculus

Let \( f \) be a continuous function on the interval \((a, x]\) then \( F \) is defined by

\[
F(x) = \int_{a}^{x} f(u) \, du.
\]

Therefore,

\[
\frac{d}{dx} F(x) = f(x).
\]

Consequently,

\[
\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x).
\]

\( F(x) \) is called the antiderivative of \( f(x) \) whereas \( f(x) \) is called the primitive function.

Example

Find the derivative of \( F(x) = \int_{2}^{x^2} \sinh^{-1} t \, dt \).

Method

Now

\[
F'(x) = \sinh^{-1} x^2 \times 2x = 2x \sinh^{-1} x^2.
\]

4.1.2 Differentiating Integrals with Parameters

Let \( f \) be a continuous function of \( x \) and \( t \) on the interval \((a, b]\) then

\[
\frac{d}{dx} \int_{a}^{b} f(x, t) \, dt = \int_{a}^{b} \frac{\partial f(x, t)}{\partial x} \, dt
\]

then

\[
\frac{d}{dx} \int_{a}^{x} f(x, t) \, dt = f(x, x) + \int_{a}^{x} \frac{\partial f(x, t)}{\partial x} \, dt.
\]

Hence,

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) \, dt = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x, t)}{\partial x} \, dt.
\]
Example

Find \( \frac{d}{dx} \left( \int_x^{x^2} \sqrt{x-t+2} \, dt \right) \).

Method

Let \( f(x, t) = \sqrt{x-t+2} \) be the integrand with \( a(x) = x \) and \( b(x) = x^2 \) then

\[
\frac{\partial f(x, t)}{\partial x} = \frac{1}{2\sqrt{x-t+2}}.
\]

Now \( f(x, a(x)) = \sqrt{2} \) and \( f(x, b(x)) = \sqrt{x-x^2+2} \). Therefore,

\[
\frac{d}{dx} \left( \int_x^{x^2} \sqrt{x-t+2} \, dt \right) = 2x\sqrt{x-x^2+2} - \sqrt{2} + \int_x^{x^2} \frac{1}{2\sqrt{x-t+2}} \, dt.
\]

Exercise 4A

1 In the following \( D \equiv \frac{d}{dx} \), and \( m, n \) are positive integers.

(a) Calculate \( D^n(x^3e^{ax}) \), \( D^4(x^2 \cos 2x) \), \( D^4(e^{2x} \cos 3x) \).

(b) Calculate \( D^m x^n \), \( n = 1, 2, 3, \ldots, m \leq n \).

Deduce that \( D^m x^n = \frac{x^{n-m}n!}{(n-m)!} \) and in particular that \( D^n x^n = n! \).

(c) Find the coefficients of \( x^n \) and \( x^{n-1} \) in

(i) \( D^n(x^2 - 1)^n \) and

(ii) \( e^x D^n(e^{-x}x^n) \).

2 Evaluate the integral

\[
u(p) = \int_0^\infty e^{-px^2} \, dx,
\]

and by differentiation with respect to \( p \), show that

\[
\int_0^\infty xe^{-px^2} \, dx = \frac{1}{p^2},
\]

\[
\int_0^\infty x^4e^{-px^2} \, dx = \frac{2}{p^3}
\]

\[
\int_0^\infty x^n e^{-px^2} \, dx = \frac{n!}{p^{n+1}}
\]

where \( n \) is a positive integer.

3 Show that the function

\[
I(a) = \int_0^\infty e^{-u^2} \cos au \, du
\]

satisfies the differential equation

\[
2 \frac{dI}{da} + aI = 0.
\]

Hence find an expression for \( I(a) \), given that \( I(0) = \frac{\sqrt{\pi}}{2} \).

4 If \( x > 0 \),

(a) evaluate

\[
\int_0^1 u^x \, du,
\]

and hence, show that

\[
\int_0^1 u^x \ln u \, du = -\frac{1}{(x+1)^2}.
\]

(b) Find an expression for

\[
\int_0^1 u^x (\ln u)^n \, du
\]

where \( n \) is a positive integer.

Can you see how a change of variable (and notation) shows that the results of questions 2 and 4 are related?

continued next page...
5 Show that the function $J(x)$, defined by the integral

$$J(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin u) \, du,$$

satisfies the differential equation (Bessel’s equation of order zero)

$$xJ'' + J' + xJ = 0.$$

is a solution of the differential equation

$$\frac{d^2y}{dx^2} + k^2y = f(x)$$

where $k$ is a constant.

6 Verify that

$$y = \frac{1}{k} \int_0^x f(u) \sin k(x - u) \, du$$

is a solution of the differential equation

$$\frac{d^2y}{dx^2} + k^2y = f(x)$$

for $x > 0$, $t > 0$.

7 By differentiating, evaluate

$$F(x, t) = \int_0^\infty e^{-xu} - e^{-tu} \, \frac{du}{u}$$

8 Solve the following integral equation by differentiating.

$$f(x) = c - \int_0^x (x - t) f(t) \, dt.$$

4.2 THE GAMMA FUNCTION

The Gamma function provides a useful means of evaluating some integrals that occur in applied mathematics. Also, it can be a useful tool for generalising and simplifying notation that occurs frequently.

4.2.1 Definition

The Gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} \, dx \quad \text{for } z > 0. \quad (\ast)$$

This integral is convergent only for $z > 0$. However, we can extend the definition of the Gamma function to incorporate all values of $z$ except for negative integers. The Gamma function is not defined for $z$ equal to a negative integer. The properties of the Gamma function are listed below.

4.2.2 Properties

If $z > 0$ then integrating by parts the equation $\ast$ we find that

$$\Gamma(z + 1) = z\Gamma(z) \quad (1)$$

and for $z$ an positive integer (ie. $z \in \mathbb{N}$) we have

$$\Gamma(z + 1) = z!$$
Rearranging equation (1) we can define the gamma function for negative values of \( z \) on the proviso that \( z \) is not a negative integer. That is, if \( z < 0 \), then for \( z \neq 0, -1, -2, \ldots \) we may define
\[
\Gamma(z) = \frac{\Gamma(z + 1)}{z}.
\]
In particular, we have that
\[
\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
\]
The graph of the Gamma function is given below.

The vertical lines on the graph of \( \Gamma(z) \) indicate where the Gamma function is not defined.

The gamma function is sometimes referred to as the generalised factorial function and has been extensively tabulated. The following table gives values of \( \Gamma(n) \) for \( n \) in the interval \([1, 2]\) in steps of 0.1 to four decimal places.

<table>
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<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
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</table>
Example

Using the given table of Gamma functions find $\Gamma(3.98)$ and $\Gamma(-2.98)$.

To find $\Gamma(3.98)$ we use the definition that $\Gamma(z+1) = z\Gamma(z)$ for $z > 0$. That is,

\[
\Gamma(3.98) = 2.98 \Gamma(2.98) = 2.98 \times 1.98 \Gamma(1.98) = 2.98 \times 1.98 \times 0.9917 \approx 5.8514.
\]

To find $\Gamma(-2.98)$ we use the definition that $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ for $z < 0$. That is,

\[
\Gamma(-2.98) = \frac{\Gamma(-1.98)}{-2.98} = \frac{1}{-2.98} \times \frac{1}{-1.98} \times \frac{1}{-0.98} \times \frac{\Gamma(1.02)}{0.02} = \frac{1}{-2.98} \times \frac{1}{-1.98} \times \frac{1}{-0.98} \times 0.9888 \approx -8.5501
\]

4.2.3 Evaluation of Integrals

Many integrals can be expressed or simplified in terms of the Gamma function. Usually, this is done by first comparing the given integral to the Gamma function integral definition then using a simple substitution to transform the given integral into the Gamma function.

Examples

Recall that

\[
\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx
\]

where $x$ is a dummy variable. That is, we can use any parameter as the integration variable.

(a) Evaluate $\int_0^\infty t^{\frac{1}{2}} e^{-t} \, dt$.

Method

Comparing the given integral to that of the Gamma function integral definition we see that

\[
z - 1 = \frac{1}{2} \quad \Rightarrow \quad z = \frac{3}{2}
\]

Therefore,

\[
\int_0^\infty t^{\frac{3}{2}} e^{-t} \, dt = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}
\]

using $\Gamma(z+1) = z\Gamma(z)$.

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}
\]

using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. 


(b) Evaluate $\int_{0}^{\infty} u^3 e^{-u^2} \, du$ in terms of the Gamma function.

Method

The integrand in our given integral has $e^{-u^2}$ as a term whereas the integrand in the Gamma function definition has $e^{-x}$ as a term. Making the substitution

$$x = u^2 \quad \text{or} \quad u = \sqrt{x}$$

will transform our integral into an integral similar to the Gamma function. Thus,

$$du = \frac{1}{2\sqrt{x}} \, dx$$

and the limits are unchanged. That is, when $u = 0$, $x = 0$ and $u = \infty$, $x = \infty$.

Hence,

$$\int_{0}^{\infty} u^3 e^{-u^2} \, du = \int_{0}^{\infty} (\sqrt{x})^3 e^{-x} \frac{dx}{2\sqrt{x}}$$

$$= \frac{1}{2} \int_{0}^{\infty} xe^{-x} \, dx.$$ Upon simplification.

Now this integral is the same as the Gamma function where $z - 1 = 1$. That is, $z = 2$. Therefore,

$$\int_{0}^{\infty} u^3 e^{-u^2} \, du = \frac{\Gamma(2)}{2} = \frac{1}{2}.$$

(c) Evaluate $\int_{0}^{\infty} e^{-t^3} \, dt$.

Method

Comparing the given integral to the Gamma function, substitute

$$u = t^3 \quad \text{or} \quad t = u^{\frac{1}{3}}.$$

Then,

$$dt = \frac{1}{3u^{\frac{2}{3}}} \, du.$$

Again the limits remain unchanged. Therefore,

$$\int_{0}^{\infty} e^{-t^3} \, dt = \int_{0}^{\infty} e^{-u} \frac{1}{3u^{\frac{2}{3}}} \, du$$

$$= \frac{1}{3} \int_{0}^{\infty} u^{-\frac{2}{3}} e^{-u} \, du.$$

This integral is now in the form of the Gamma function where $z - 1 = -\frac{2}{3}$. That is, $z = \frac{1}{3}$. Therefore,

$$\int_{0}^{\infty} e^{-t^3} \, dt = \frac{\Gamma\left(\frac{4}{3}\right)}{3}.$$
**Exercise 4B**

1. Evaluate each of the following.
   
   (a) $\Gamma(5)$
   
   (b) $\Gamma \left( \frac{7}{2} \right)$
   
   (c) $\Gamma \left( -\frac{1}{2} \right)$
   
   (d) $\Gamma \left( -\frac{5}{2} \right)$

2. By first making a substitution, evaluate each of the following integrals in terms of the gamma function.

   (a) $\int_1^\infty e^{-x} dx$
   
   (b) $\int_1^\infty e^{x^2} dx$
   
   (c) $\int_1^\infty \frac{1}{x} \Gamma(z) dx$

3. Prove the following results.

   (a) $\Gamma \left( \frac{1}{2} \right) = \sqrt{\pi}$
   
   (b) $\Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2}$

---

**4.3 THE BETA FUNCTION**

**4.3.1 Definition**

The Beta function is defined as

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m > 0, n > 0.$$  

Sometimes the Beta function is written as $B_{m,n}$.

**4.3.2 Properties**

$$B(m, n) = B(n, m),$$

$$B(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy,$$

$$B(m, n) = 2 \int_0^{\pi/2} \cos^{(2m-1)} \theta \sin^{(2n-1)} \theta d\theta,$$

and

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

**4.3.3 Evaluation of Integrals**

Since the Beta function has several forms, many integrals can be expressed in terms of this function. This is usually done by a simple substitution.

**Examples**

Recall the Beta function definition.

(a) Evaluate $\int_0^1 \sqrt{\frac{t}{1-t}} dt$ in terms of the Beta function.
Method

Now reshaping the given integral we have that

$$\int_{0}^{1} \sqrt{\frac{t}{1-t}} \, dt = \int_{0}^{1} t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}} \, dt.$$  

On comparison with the Beta function, we find that

$$m - 1 = \frac{1}{2} \quad \Rightarrow \quad m = \frac{3}{2},$$

$$n - 1 = -\frac{1}{2} \quad \Rightarrow \quad n = \frac{1}{2}.$$  

Therefore,

$$\int_{0}^{1} \sqrt{\frac{t}{1-t}} \, dt = B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) = \frac{\pi}{2} \quad \text{using } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$  

(b) Evaluate $\int_{0}^{\frac{1}{2}} u^2(1-2u) \, du$ using the Beta function.

Method

The given integral is not in the form of the Beta function. That is, the integrand does not look like the integrand of the Beta function nor does the limits of the integral. However, by a simple change of variables we find that the given integral can be transformed into the Beta function.

Let

$$x = 2u \quad \text{or} \quad u = \frac{x}{2}.$$  

We choose this substitution so the $1-2u \rightarrow 1-x$, which is comparable to the Beta function.

Using this substitution,

$$du = \frac{dx}{2}.$$  

We also need to change the limits from $u$ limits (the original limits) to $x$ limits, as our substitution changes our integral from a $u$ integral to an $x$ integral. Therefore,

when $u = 0$, $x = 0$ and when $u = \frac{1}{2}$, $x = 1$.

Hence, the integral becomes

$$\int_{0}^{\frac{1}{2}} u^2(1-2u) \, du = \int_{0}^{1} \left(\frac{x}{2}\right)^2 (1-x) \frac{dx}{2}$$

$$= \frac{1}{2\pi} \int_{0}^{1} x^2(1-x) \, dx$$  

The integral is now in the form of the Beta function.
Therefore, \[ m - 1 = 2 \quad \Rightarrow \quad m = 3 \]
\[ n - 1 = 1 \quad \Rightarrow \quad n = 2. \]

Now
\[ \int_0^1 u^2(1 - 2u) \, du = \frac{1}{2^2} B(3, 2) \]
\[ = \frac{\Gamma(3) \Gamma(2)}{8\Gamma(5)} \]
\[ = \frac{1}{44!} \]
\[ = \frac{1}{96}. \]

**Exercise 4C**

1. Evaluate the following integrals in terms of the Gamma function.
   
   (a) \[ \int_0^\infty x^7 e^{-x^5} \, dx \]
   
   (b) \[ \int_0^1 x^2 (1 - x^{2/3}) \, dx \]
   
   (c) \[ \int_0^\pi/2 \cos^3 t \sin^7 t \, dt \]
   
   (d) \[ \int_0^\pi/2 \sqrt{\tan \theta} \, d\theta \]
   
   (e) \[ \int_0^\infty \frac{y^6}{(1 + y)^{10}} \, dy \]
   
   (f) \[ \int_0^1 \left( \ln \frac{1}{x} \right)^{2/3} \, dx \]
   
   (g) \[ \int_0^\infty e^{-x^2} \, dx \]
   
   (h) \[ \int_0^\infty e^{-x} \sqrt{x} \, dx \]
   
   (i) \[ \int_0^\infty (1 + x^2)^2 e^{-x} \, dx \]
   
   (j) \[ \int_0^1 \frac{dx}{\sqrt{1 - x^3}} \]
   
   (k) \[ \int_0^\pi \sin^6 \theta \cos^4 \theta \, d\theta \]
   
   (l) \[ \int_0^1 \frac{1 - x^2}{1 + x^2} \, dx \]

2. Prove the following results.

   (a) \[ B(m, n) = 2 \int_0^1 x^{2m-1} (1 - x^2)^{n-1} \, dx \]
   
   (b) \[ B(m, n) = \frac{1}{(b-a)^{m+n-1}} \int_a^b (b-x)^{m-1} (x-a)^{n-1} \, dx \]
   
   (c) \[ B\left( p + \frac{1}{2}, q + \frac{1}{2} \right) = 2 \int_0^{\pi/2} \cos^p \theta \sin^q \theta \, d\theta. \]
   
   (d) \[ B(m, n) = \int_0^\infty \frac{t^{n-1}}{(1 + t)^{m+n}} \, dt \]
   
   \[ = \int_0^\infty \frac{t^{m-1}}{(1 + t)^{m+n}} \, dt \]
   
   continued next page...
and hence,

\( B(m, n) = \frac{1}{2} \int_0^\infty \frac{t^{n-1} + t^{m-1}}{(1 + t)^{m+n}} \, dt \)

\( = \int_0^\infty \frac{t^{m-1}}{(1 + t)^{m+n}} \, dt. \)

(e) \( B(m, 1) = \frac{1}{m} \)

(f) \( B(m + 1, n) + B(m, n + 1) = B(m, n). \)

4.4 THE ERROR FUNCTION

4.4.1 Definition

The error function is defined by

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du. \]

This function occurs frequently in heat and probability problems as well as fluid mechanics.

4.4.2 Properties

(a) \( \text{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-u^2} \, du \)

\( = -\text{erf}(x), \)

(b) \( \text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \, du \)

\( = 1. \)

(c) \( \text{erc}(x) = e^{x^2} \text{erfc}(x) \)

Graphically, the error function is the area under the normal distribution curve \( e^{-x^2}. \)

4.4.3 Complimentary Error Function

The complementary error function is defined by

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} \, du. \]
Consequently,
\[ \text{erfc}(x) = 1 - \text{erf}(x). \]
That is,
\[ \text{erf}(x) + \text{erfc}(x) = 1. \]

### 4.4.4 Evaluation of Integrals

Many integrals that cannot be evaluated analytically can be transformed into the error function.

**Examples**

Recall the error function definition. That is,
\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du. \]

(a) Evaluate \( \int_0^\infty e^{-4t^2} \, dt \).

**Method**

Compare the given integral to the error function definition. Then
\[
\int_0^\infty e^{-4t^2} \, dt = \frac{1}{2} \int_0^\infty e^{-u^2} \, du \quad \text{where } u = 2t \text{ and } du = 2dt.
\]
\[
= \frac{1}{2} \times \left( \frac{\sqrt{\pi}}{2} \times \frac{2}{\sqrt{\pi}} \right) \int_0^\infty e^{-u^2} \, du \quad \text{using the error function definition}
\]
\[
= \frac{\sqrt{\pi}}{4} \text{erf}(\infty)
\]
\[
= \frac{\sqrt{\pi}}{4} \quad \text{where erf}(\infty) = 1.
\]

(b) Evaluate \( \int_0^3 e^{-w^2+2w-1} \, dw \).

**Method**

Compare the given integral to the error function definition. Then
\[
\int_0^3 e^{-w^2+2w-1} \, dw = \int_0^3 e^{-(w^2-2w+1)} \, dw
\]
\[
= \int_0^3 e^{-(w-1)^2} \, dw \quad \text{completing the square}
\]
\[
= \int_{-1}^2 e^{-u^2} \, du \quad \text{where } u = w - 1 \text{ and } du = dw.
\]
\[
= \frac{\sqrt{\pi}}{2} \times \frac{2}{\sqrt{\pi}} \int_{-1}^2 e^{-u^2} \, du
\]
\[
= \frac{\sqrt{\pi}}{2} (\text{erf}(2) - \text{erf}(-1))
\]
\[
= \frac{\sqrt{\pi}}{2} (\text{erf}(2) + \text{erf}(1)) \quad \text{where erf}(-x) = -\text{erf}(x).
(c) Simplify \( \int_0^\infty e^{-2st} \text{erf}(t) \, dt \).

Method

To simplify this integral we have to use the integral definition of the error function and our knowledge of reversing the order of integration. That is,

\[
\int_0^\infty e^{-2st} \text{erf}(t) \, dt = \int_0^\infty e^{-2st} \left\{ \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} \, du \right\} \, dt
\]

\[
= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-2st} \int_0^t e^{-u^2} \, du \, dt
\]

\[
= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^t e^{-2st} e^{-u^2} \, du \, dt
\]

To simplify this integral we have to reverse the order of integration. This means that we have to find the region of integration. We look at the limits and find that the inner limit:

\[ u \text{ goes from } u = 0 \text{ to } u = t \]

and the outer limit:

\[ t \text{ goes from } t = 0 \text{ to } t = \infty. \]

Therefore, the shaded region in the figure below is the region of integration.

Now reversing the order of integration we integrate with respect to ‘t’ first. This means that we need to find the new limits of integration. Now \( t \) limits are the new inner limits and \( u \) limits are the new outer limits. (Note: we can not simply interchange the integrals around. See Chapter 1). Therefore,

\[ t \text{ goes from } t = u \text{ to } t = \infty \]

and

\[ u \text{ goes from } u = 0 \text{ to } u = \infty \]

to completely define the region of integration. Therefore,

\[
\frac{2}{\sqrt{\pi}} \int_0^\infty \int_u^t e^{-2st} e^{-u^2} \, du \, dt = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_u^\infty e^{-u^2} e^{-2st} \, dt \, du
\]

\[
= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left[ e^{-2st} \right]_u^\infty \, du
\]

\[
= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left( e^{-2su} - e^{-2su} \right) \, du \quad \text{as } e^{-2su} \to 0.
\]
Therefore,
\[
\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^t e^{-2st} e^{-u^2} du \, dt = \frac{1}{s\sqrt{\pi}} \int_0^\infty e^{-(u^2+2su)} \, du \quad \text{completing the square}
\]
\[
= \frac{1}{s\sqrt{\pi}} \int_0^\infty e^{-(u+s)^2+s^2} \, du \quad \text{substituting } v = u + s \text{ then } dv = ds
\]

Hence,
\[
\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^t e^{-2st} e^{-u^2} du \, dt = \frac{e^s}{2s} \times 2 \int_s^\infty e^{-v^2} \, dv \quad \text{using the defintion for } \text{erf}(x))
\]
\[
= \frac{e^s}{2s} (\text{erf}(\infty) - \text{erf}(s))
\]
\[
= \frac{e^s}{2s} (1 - \text{erf}(s)) \quad \text{as } \text{erf}(\infty) = 1.
\]

### 4.5 BESSEL FUNCTIONS

#### 4.5.1 The Bessel Equation

Bessel’s differential equation of order $\nu$ is

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0.
\]

When $\nu$ is not an integer or zero, the method of solution yields two linear independent solutions, namely,

\[
J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{r=0}^\infty \frac{(-1)^r}{\Gamma(\nu + r + 1)} \left(\frac{x}{2}\right)^{2r}
\]

and

\[
J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{r=0}^\infty \frac{(-1)^r}{\Gamma(-\nu + r + 1)} \left(\frac{x}{2}\right)^{2r}
\]

$J_\nu(x)$ and $J_{-\nu}(x)$ are called Bessel functions of order $\nu$ and $-\nu$ respectively. These functions are called Bessel functions of the first kind.

Hence, the general solution to the Bessel’s equation when $\nu$ is not an integer or zero is

\[
y = AJ_\nu(x) + BJ_{-\nu}(x)
\]

where $A$ and $B$ are arbitrary constants.

If $\nu$ is an integer or zero ($\nu = n$ say), then the general solution to the Bessel’s equation is

\[
y = AJ_n(x) + BY_n(x)
\]

where $Y_n(x)$ is called the Bessel function of the second kind or order $\nu$. 
4.5.2 Special Case

If \( \nu = 0 \) the solution of the Bessel’s equation is

\[
y(x) = J_0(x) + BY_0(x)
\]

where \( J_0(x) \) is the Bessel function of the first kind and \( Y_0(x) \) is the Bessel function of the second kind and is defined as

\[
Y_0(x) = J_0(x) \ln x - \sum_{r=1}^{\infty} \frac{(-1)^r}{(r!)^2} \left( \frac{x}{2} \right)^{2r} \phi(r)
\]

where \( \phi(r) = \sum_{s=1}^{r} \frac{1}{s} \).

The graph of these two functions is given below.

Particular note should be given to the behaviour of both of these functions as \( x \to 0 \) and \( x \to \infty \). It can be seen from the graphs that \( J_0(x) \) tends towards 1 and \( Y_0(x) \) tends towards \( -\infty \) as \( x \to 0 \). Also, both functions are oscillatory in nature as \( x \to \infty \). Both \( J_0(x) \) and \( Y_0(x) \) tend towards 0 as \( x \to \infty \).

If \( \nu = 1 \) the solution of the Bessel’s equation is

\[
y(x) = J_1(x) + BY_1(x)
\]

where \( J_1(x) \) is the Bessel function of the first kind and \( Y_1(x) \) is the Bessel function of the second kind.

The graphs of these function are in the following figures.

From the graphs it can be seen that \( J_1(x) \) and \( Y_1(x) \) have similar behaviour to the Bessel functions of order zero except that \( J_1(x) \) tends toward 0 as \( x \to 0 \).
4.5.3 Properties

Many recurrence relations exist for $J_\nu(x)$. For example,

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

The orthogonality condition for the Bessel Functions on the interval $(0,l)$ is

$$\int_0^l x J_\nu(k_1x) J_\nu(k_2x) dx = 0$$

where the values of $k_1$ and $k_2$ are obtained from finding the zeros $\{ k \}$ of

$$J_\nu(kl) = 0.$$

There are many other properties of the Bessel function that will not be covered in this course.

4.6 STEP (HEAVISIDE) FUNCTION

4.6.1 Definition

The Step or Heaviside function is defined by

$$h(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$

Some books identify this function with a capital $H$.

Graphically, $h(t)$ (where $a = 0$) has the appearance

![Graph of the step function](image)

Example

(a) Sketch the graph of $h(t+1)$.

Method

Recall the step function definition then

$$h(t+1) = \begin{cases} 1 & t \geq -1 \\ 0 & t < -1 \end{cases}.$$
Therefore, the graph of \( h(t + 1) \) is

(b) Sketch the graph of \( h(t + 1) \cos t \).

Method

Recall that

\[
h(t + 1) = \begin{cases} 
1 & t \geq -1 \\
0 & t < -1 
\end{cases}
\]

Then

\[
h(t + 1) \cos t = \begin{cases} 
1 \times \cos t & t \geq -1 \\
0 \times \cos t & t < -1 
\end{cases}
\]

Hence,

\[
h(t + 1) \cos t = \begin{cases} 
\cos t & t \geq -1 \\
0 & t < -1 
\end{cases}
\]

Therefore, the graph of \( h(t + 1) \cos t \) is

4.6.2 Integrals using the Step Function

At times we may need to evaluate integrals that involve the step function. When this is the case we simply apply the step function definition. In other words, we simply break the integral up into respective intervals.

Example

Evaluate \( \int_{-\infty}^{\infty} h(t - 3)e^{-2t} \, dt \).
Method

\[ \int_{-\infty}^{\infty} h(t-3)e^{-2t} \, dt = \int_{3}^{\infty} 1 \times e^{-2t} \, dt \quad \text{using the step function definition} \]

\[ = \left. \frac{e^{-2t}}{-2} \right|_{3}^{\infty} \]

\[ = \frac{e^{-6}}{2}. \]

This function can be compared to a power switch that was off and then turned on.

### 4.7 DIRAC DELTA (IMPULSE) FUNCTION

#### 4.7.1 Definition

In various applications we find that we need to mathematically represent the action of large forces over short intervals of time. For instance, the action of a tennis ball being hit, a ship that is hit by a high single wave, the detonation of bomb or the voltage surge in an electric current. These actions can be best described by the following function.

Consider the function

\[ f_k(t-a) = \begin{cases} \frac{1}{k} & a < t < a+k, \\ 0 & \text{otherwise} \end{cases} \]

then

\[ \int_{0}^{\infty} f_k(t-a) \, dt = \int_{a}^{a+k} \frac{1}{k} \, dt \]

\[ = 1. \]

Graphically,

As \( k \to 0 \) we find that \( t \to a \). Then we say that

\[ \lim_{k \to 0} f_k(t-a) = \delta(t-a) \]

where \( \delta(t) \) is called the Dirac Delta function. Alternatively, the Dirac delta function is defined by

\[ \delta(t-a) = 0, \quad t \neq a \]

and

\[ \int_{-\infty}^{\infty} \delta(t-a) \, dt = 1. \]
4.7.2 Properties

\[ \int_0^\infty \delta(t - a) \, dt = 1, \quad \text{for } a > 0 \]

\[ \int_{-\infty}^\infty f(t) \delta(t - a) \, dt = f(a), \]

\[ \int_0^\infty f(t) \delta(t - a) \, dt = f(a), \quad \text{for } a > 0 \]

\[ \int_{-\infty}^x \delta(t) \, dt = h(x). \]

**Note:**

Recall the property

\[ \int_{-\infty}^x \delta(t) \, dt = h(x) \]

where \( h(x) \) is the step function. Using the Fundamental Theorem of Calculus, we know that

\[ \delta(x) = h'(x). \]

The Dirac Delta function can be pictured as below

![Diagram of Dirac Delta function](image)

This function is sometimes referred to the impulse function and therefore used to mathematically explain impulse forces. By an impulsive force, it is meant that a force has a very large amplitude but acts over a very small period of time. Here, the source of the impulse is at \( t = a \).

**Exercise 4D**

1. From the definition, prove the following properties and results.
   - (a) \( \text{erf}(-x) = -\text{erf}(x) \)
   - (b) \( \text{erf}(\infty) = 1 \)
   - (c) \( \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du = \frac{1}{a} \text{erf}(ax) \)
   - (c) \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \left\{ x - \frac{x^3}{3} + \frac{1}{2!} \frac{x^5}{5} - \frac{1}{3!} \frac{x^7}{7} + \cdots \right\} \)

2. Evaluate \( \int_0^\infty \text{erfc}(x) \, dx \).

3. Let \( f(t) \) be given by
   - (i) \( h(t)h(1-t) \)
   - (ii) \( h(t) + h(t-1) \)
(iii) \( \sin b(t - \pi) \)

(iv) \( \cos t(h(t - \pi) - h(t)) \)

(v) \( \sin(t - \pi)h(t - \pi) \)

For each \( f(t) \)

(a) Sketch the graph of the function.

(b) Evaluate

\[ \int_0^\infty f(t)e^{-pt} \, dt. \]

4 Evaluate the following integrals:

(a) \( \int_0^\infty h(t - 2)e^{-pt} \, dt \)

(b) \( \int_0^\infty h(t + 2)e^{-pt} \, dt \)

(c) \( \int_0^\infty \delta(t - 2)e^{-pt} \, dt \)

(d) \( \int_0^\infty (h(t - 2) + h(t))e^{-pt} \, dt \)

(e) \( \int_0^\infty \delta(t - 1)\sin(3t)e^{-pt} \, dt \)

5 Interpret the following graphs of \( f(t) \) in terms of the step function and hence, evaluate

\[ \int_0^\infty f(t)e^{-pt} \, dt. \]

(a) \[ f(t) \]

(b) \[ f(t) \]

6 Show that

\[ \int_0^\infty e^{-px}f(x - a)h(x - a) \, dx = e^{-ap}F(p) \]

where \( F(s) = \int_0^\infty f(x)e^{-ps} \, dx. \)

7 Solve the equation

\[ y'(x) - y \tan x = 2 \sin x h(x). \]

8 Evaluate \( \int_0^\infty ze^{-z^2}\text{erf}(z) \, dz. \)

9 Show that \( \text{erc}(x) \) satisfies the differential equation

\[ \frac{dy}{dx} = 2xy - \frac{2}{\sqrt{\pi}} \]

with \( y = 1 \) when \( x = 0 \).

10 If \( I(a) = \int_0^\infty \frac{e^{-ax^2}}{1 + x^2} \, dx \), show that

\[ I'(a) - I(a) = -\frac{1}{2} \sqrt{\frac{\pi}{a}} \]

and hence that

\[ I(a) = \frac{1}{2}\pi\text{erc}(\sqrt{a}). \]

11 Let \( g(t) \) be defined as

\[ g(t) = \text{rect} \left( \frac{t}{T} - 0.5 \right) \\
= h(t) - h(t - T) \]
where \( t \geq 0 \) and \( T \) is a constant.

(a) Sketch \( g(t) \).

(b) Let \( k(t) = g(t) \) then sketch \( k(t-T-u) \) with respect to \( u \).

(c) Let \( w(u) = k(t-u)g(u) \). Sketch \( w(u) \) if \( t \leq T \), then if \( t > T \).

(d) Find \( f(t) \), if

\[
 f(t) = g(t) * k(t) = \int_0^t g(u)k(t-u) \, du.
\]

\( f(t) \) is known as \( \text{tri}\left(\frac{t}{T} - 1\right) \).

(e) Hence, sketch \( f(t) \).

* This is a simple Nyquist system with rectangular filters.

12 Let \( g(t) \) be defined as

\[
 g(t) = \text{rect}\left(\frac{t - 0.75T}{1.5T}\right),
\]

\[
 k(t) = \text{rect}\left(\frac{t - 1.25T}{1.5T}\right).
\]

13 Evaluate the following integrals:

(a) \( \int_a^b \delta(t) \, dt \)

(b) \( \int_a^b \delta'(t) \, dt \)
Chapter 5: Laplace Transforms

The Laplace Transform is a useful tool that is used to solve many mathematical and applied problems. In particular, the Laplace transform is a technique that can be used to solve linear constant coefficient differential equations in one or more equations. This tool is used to transform the given system of differential equations into a system of algebraic equations that are then solved simultaneously. The algebraic system obtained is usually much easier to solve than the given differential equations. Graphically,

As shown in the Figure (a) the original function $f(t)$ (solution to the given differential equation) is transformed into a new function $F(s)$ via a transfer function. This function $F(s)$ is obtained from solving an algebraic equation in $s$. Once $F(s)$ is found we use transforms to obtain the original variable $t$ and hence, obtain the solution to the given differential equation. This is described in Figure (b).

The Laplace transform method is used in a wide number of engineering problems, in particular, the design of control systems in electrical engineering or the vibration of mass-spring system to a unit impulse. For example, Laplace transforms can be used to determine the response of a damped mass-spring system which is initially at rest and is suddenly given a sharp hammer blow. The governing equations could be represented by

$$y'' + 3y' + 2y = \delta(t - a)$$

where

$$y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$ 

Our usual methods cannot solve this equation due to the impulse function, $\delta(t - a)$. However, using Laplace transforms we can solve this equation. Its solution being

$$y(t) = \left(e^{-(t-a)} - e^{-2(t-a)}\right)h(t - a)$$

where $h(t)$ is the step function.

Discussion on solving differential equations will be given in later in this chapter. However, first, we need to become familiar with the Laplace transform and its properties.
5.1 DEFINITION

Let \( f(t) \) be a function defined on the interval \([0, \infty)\). Then the Laplace Transform of a function \( f(t) \) is defined by

\[
\mathcal{L}\{f(t)\} = F(p) = \int_0^\infty e^{-pt} f(t) \, dt
\]

**Note:**

Here the independent variable of \( f \) is \( t \) and the transfer parameter is \( p \). This means we are transferring from the \( t \) domain to the \( p \) domain when using Laplace transforms. The parameter \( p \) can be complex. However, provided \( f(t) \) is a ‘well behaved’ function, then the \( \text{Re}\{p\} \) must be positive for the above integral to converge.

In some texts \( s \) is used as the transfer parameter instead of \( p \) although the same definition holds. That is,

\[
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \, dt.
\]

In MATH202 both parameters may be used.

There are a variety of transform functions which can aid the mathematician or engineer to solve problems. These will not be discussed here. However, it is worth mentioning two of these transforms.

**Fourier:**

\[
F\{f(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} f(t) \, dt
\]

**Two-sided Laplace:**

\[
\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-pt} f(t) \, dt
\]

5.2 LAPLACE TRANSFORM OF FUNCTIONS

Let \( \mathcal{L}\{f(t)\} = F(p) \) and \( \mathcal{L}\{g(t)\} = G(p) \) then particular results for the Laplace transform are

\[
\mathcal{L}\{\alpha f(t)\} = \alpha \mathcal{L}\{f(t)\} \quad \text{where } \alpha \text{ a constant.}
\]

\[
\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} \quad \text{where } \alpha \text{ and } \beta \text{ are constants.}
\]

\[
\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{p}{a}\right)
\]

\[
\mathcal{L}\{e^{-at} f(t)\} = F(p + a) \quad \text{(Shift Theorem)}
\]

\[
\mathcal{L}\{f(t-a) h(t-a)\} = e^{-ap} F(p) \quad \text{(Second Shift Theorem)}
\]

\[
\mathcal{L}\{f(t) h(t-a)\} = e^{-ap} \mathcal{L}\{f(t+a)\} \quad \text{(Third Shift Theorem)}
\]

\[
\mathcal{L}\{tf(t)\} = -\frac{d}{dp} F(p).
\]

where \( a \) is a constant.
Examples

Recall that
\[ \mathcal{L} \{ f(t) \} = F(p) = \int_0^\infty f(t) e^{-pt} \, dt. \]

(a) Find the Laplace transform of \( f(t) = t \).

Method

\[
\mathcal{L} \{ f(t) \} = \mathcal{L} \{ t \} = \int_0^\infty t e^{-pt} \, dt \quad \text{replacing } f(t) \text{ by } t
\]
\[
= \frac{1}{p} \int_0^\infty e^{-pt} \, dt \quad \text{using integration by parts}
\]
\[
= \frac{1}{p^2}.
\]

Note:
Here \( f \) is a function of the independent variable \( t \) therefore we found the Laplace transform of \( f \) with respect to the independent variable \( t \). However, not all functions will have the independent variable being \( t \). For instance, the independent variable for \( f(x) \) is \( x \). If this is the case then we simply take Laplace transforms with respect to \( x \). In all cases, the Laplace transform function \( F(p) \) will be the same.

Convention

We usually use capital letters to represent the Laplace transform of a function. For example,

\[
\mathcal{L} \{ f(x) \} = F(p), \quad \mathcal{L} \{ g(u) \} = G(p) \quad \text{and} \quad \mathcal{L} \{ z(w) \} = Z(p).
\]

For example,
If \( f(x) = x \), say, then \( \mathcal{L} \{ f(x) \} = \mathcal{L} \{ x \} = \frac{1}{p^2} \) or if \( g(u) = u \) then \( \mathcal{L} \{ g(u) \} = \mathcal{L} \{ u \} = \frac{1}{p^2} \).

A graph of the function \( f(x) = x \) and its transform (ie \( \mathcal{L} \{ f(x) = x \} = F(p) = \frac{1}{p^2} \)) is shown below.
(b) Find the Laplace transform of $f(t) = \sin t$.

**Method**

The independent variable is $t$. Therefore, find the Laplace transform of $f(t)$ with respect to $t$. That is,

$$
\mathcal{L}\{\sin t\} = \int_0^\infty \sin t \, e^{-pt} \, dt \quad \text{replacing } f(t) \text{ by } \sin t.
$$

$$
= \frac{1}{s} \int_0^\infty \cos t \, e^{-pt} \, dt \quad \text{using integration by parts}
$$

$$
= \frac{1}{p^2 + 1}.
$$

**Note:** Evaluation of this integral could have been done by Integral [31].

A graph of $f(t) = \sin t$ and its transform $F(p)$ is shown below.

Similarly, the $\mathcal{L}\{\cos t\}$ is

$$
\mathcal{L}\{\cos t\} = F(p) = \int_0^\infty \cos t \, e^{-pt} \, dt \quad \text{replacing } f(t) \text{ by } \cos t.
$$

$$
= \frac{p}{p^2 + 1} \quad \text{using integration by parts.}
$$

A graph of $f(t) = \cos t$ and its transform $F(p)$ is shown below.
(c) Find the Laplace transform of \( f(t) = t + \sin t \).

**Method**

Recall that \( \mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \). Let \( f(t) = t \) and \( g(t) = \sin t \) then

\[
\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{t + \sin t\} \\
= \mathcal{L}\{t\} + \mathcal{L}\{\sin t\} \\
= \frac{1}{p^2} + \frac{1}{p^2 + 1}.
\]

(d) Find the Laplace transform of

\[
f(t) = \begin{cases} 
1, & t \geq 3 \\
0, & t < 3
\end{cases}.
\]

**Method**

Recall that

\[
\mathcal{L}\{f(t)\} = F(p) = \int_0^\infty f(t) e^{-pt} dt.
\]

Therefore,

\[
\mathcal{L}\{f(t)\} = \int_0^3 f(t) e^{-pt} dt + \int_3^\infty f(t) e^{-pt} dt \\
= \int_3^\infty e^{-pt} dt \quad \text{replacing } f(t) \text{ by its definition.} \\
= \left[ \frac{e^{-pt}}{-p} \right]_3^\infty = \frac{e^{-3p}}{p}.
\]

### 5.2.1 Shift Theorem

The Shift theorem is an important theorem and will be used quite frequently throughout this chapter.

Recall that

\[
\mathcal{L}\{e^{-at}f(t)\} = F(p + a)
\]

where \( a \) is an arbitrary constant.

This theorem means that if we want to find \( \mathcal{L}\{e^{-at}f(t)\} \) we simply find the Laplace transform of \( f(t) \) which is \( F(p) \) and then replace \( p \) by \( p + a \).

Graphically, this theorem is represented by \( F(p) \) shifted to the right or left depending on the sign of \( a \).
Mathematically, we can write that
\[
\mathcal{L} \{ e^{-at} f(t) \} = F(p + a) = F(p)|_{p \rightarrow p+a}.
\]

Also, we shall use the shift theorem in Section 5.4 to find the inverse Laplace transform of a given function of \( p \).

**Examples**

(a) Find the Laplace transform of \( f(t) = e^{-2t} \sin t \).

**Method**

\[
\mathcal{L} \{ e^{-2t} \sin t \} = \int_0^\infty \left( e^{-2t} \sin t \right) e^{-pt} dt \quad \text{replacing } f(t) \text{ by } e^{-2t} \sin t
\]
\[
= \int_0^\infty \sin t \ e^{-(2+p)t} dt \quad \text{integrating by parts}
\]
\[
= \frac{1}{(p + 2)^2 + 1}.
\]

Alternatively, we can use the shift theorem. That is,
\[
\mathcal{L} \{ e^{-at} f(t) \} = F(p + a) \quad \text{where } F(p) = \mathcal{L} \{ f(t) \}.
\]

In our example, we see that \( f(t) = \sin t \) and \( a = 2 \) then
\[
F(p) = \mathcal{L} \{ f(t) \} = \frac{1}{p^2 + 1} \quad \text{from example (d)}.
\]

Therefore,
\[
F(p + 2) = \frac{1}{(p + 2)^2 + 1} \quad \text{where } a = 2
\]
\[
= \frac{1}{(p + 2)^2 + 1}.
\]

Hence,
\[
\mathcal{L} \{ e^{-2t} \sin t \} = \frac{1}{(p + 2)^2 + 1}.
\]

Below is a graph of the function \( f(t) = e^{-2t} \sin t \) and the transform function \( F(p + 2) = \frac{1}{(p + 2)^2 + 1} \).
(b) Find the Laplace transform of \( e^{3t} \cos 4t \).

Method

\[
\mathcal{L} \{ e^{3t} \cos 4t \} = \int_0^\infty e^{3t} \cos 4t \ e^{-pt} \, dt \quad \text{replacing } f(t) \text{ by } e^{3t} \cos 4t
\]

\[
= \int_0^\infty \cos(4t) \ e^{-(p-3)t} \, dt \quad \text{integrating by parts}
\]

\[
= \frac{p - 3}{(p - 3)^2 + 16}.
\]

Alternatively, using the shift theorem we let \( f(t) = \cos 4t \) where \( a = -3 \) then

\[
F(p) = \frac{p}{p^2 + 16} \quad \text{from Laplace tables where } n = 4.
\]

Hence,

\[
F(p - 3) = \frac{p - 3}{(p - 3)^2 + 16} \quad \text{where } a = -3.
\]

Therefore,

\[
\mathcal{L} \{ e^{3t} \cos 4t \} = \frac{p - 3}{(p - 3)^2 + 16}.
\]

The previous two examples show alternative methods of finding the Laplace transform of functions of the form \( \mathcal{L} \{ e^{-at} f(t) \} \). Each of these methods is acceptable. However, time and lengthy integration is reduced if the shift theorem and the Laplace transform tables are used.

5.2.2 The Convolution Theorem

The Convolution Theorem, states

\[
\mathcal{L} \left\{ \int_0^t f(t-u) \ g(u) \, du \right\} = F(p) \ G(p)
\]

and is often written as

\[
\mathcal{L} \{ f \ast g \} = F \cdot G
\]

where \( \mathcal{L} \{ f(t) \} = F(p) \) and \( \mathcal{L} \{ g(t) \} = G(p) \).

Alternatively, the Convolution theorem can also be written in the form

\[
\mathcal{L} \left\{ \int_0^t f(u) \ g(t-u) \, du \right\} = F(p) \ G(p)
\]

where the end result is still the same.

Note:

It should be noted that the given \( \int_0^t f(u) \ g(t-u) \, du \) is a function of \( t \) and therefore, when we take Laplace transforms we are doing it with respect to \( t \).

A special case of the Convolution Theorem is the result for the Laplace transform of an integral.

\[
\mathcal{L} \left\{ \int_0^t f(u) \, du \right\} = \frac{1}{p} F(p).
\]
Examples

(a) Find the Laplace transform of \( \int_0^t u \sin(t-u) \, du \).

Method

It can be easily seen that \( \int_0^t u \sin(t-u) \, du \) is a function of \( t \). Therefore, let

\[ k(t) = \int_0^t u \sin(t-u) \, du. \]

Therefore,

\[ \mathcal{L}\{k(t)\} = \int_0^\infty k(t) e^{-pt} \, dt \]

\[ = \int_0^\infty \left( \int_0^t u \sin(t-u) \, du \right) e^{-pt} \, dt. \]

Reversing the order of integration.

That is,

\[ \mathcal{L}\{k(t)\} = \int_0^\infty u \left( \int_u^\infty e^{-pt} \sin(t-u) \, dt \right) \, du \]

as \( u \) is a constant in the inner integral

Let \( z = t-u \) then \( dz = dt \);

when \( t = u, \quad z = 0; \quad t \to \infty, \quad z \to \infty \).

\[ = \int_0^\infty ue^{-u} \left( \int_0^\infty e^{-pz} \sin(z) \, dz \right) \, du \]

The inner integral is \( \mathcal{L}\{\sin z\} \). That is,

\[ \mathcal{L}\{\sin z\} = \frac{1}{p^2 + 1}. \]

\[ \frac{1}{p^2 + 1} \int_0^\infty ue^{-u} \, du \]

This is the \( \mathcal{L}\{u\} \) which is \( \frac{\Gamma(2)}{p^2} \).

\[ = \frac{1}{p^2(p^2 + 1)}. \]

Alternatively, use the Convolution theorem and let

\[ f(t-u) = \sin(t-u) \quad \text{and} \quad g(u) = u. \]

Thus,

\[ f(t) = \sin t \quad \text{(replacing \( t-u \) by \( t \)) \quad \text{and} \quad g(t) = t.} \]

Taking Laplace transforms of both \( f(t) \) and \( g(t) \) with respect to \( t \) we have

\[ \mathcal{L}\{f(t)\} = F(p) = \frac{1}{p^2 + 1} \quad \text{and} \quad \mathcal{L}\{g(t)\} = G(p) = \frac{1}{p^2}. \]

Therefore,

\[ \mathcal{L}\left\{ \int_0^t u \sin(t-u) \, du \right\} = \frac{1}{p^2 + 1} \times \frac{1}{p^2} \]

\[ = \frac{1}{p^2(p^2 + 1)}. \]
Note:

It can be seen in this example that using the Convolution theorem is quicker than doing the double integration. Therefore, it wise that students learn this theorem.

(b) Find \( \mathcal{L} \left\{ \int_0^t e^{-2u} \sinh 3(t - u) \, du \right\} . \)

**Method**

Matching the functions in the Convolution theorem we can see that

\[
f(t - u) = \sinh 3(t - u) \quad \text{and} \quad g(u) = e^{-2u}
\]

Thus,

\[
f(t) = \sinh 3t \quad \text{(replacing} \ t - u \ \text{by} \ t) \quad \text{and} \quad g(t) = e^{-2t}.
\]

We take Laplace transforms of both \( f(t) \) and \( g(t) \) with respect to \( t \). Therefore, using the Laplace transform tables, we find that

\[
\mathcal{L} \{ f(t) \} = F(p) = \frac{3}{p^2 - 9} \quad \text{and} \quad \mathcal{L} \{ g(t) \} = G(p) = \frac{1}{p + 2}.
\]

Therefore,

\[
\mathcal{L} \left\{ \int_0^t e^{-2u} \sinh(3(t - u)) \, du \right\} = \frac{3}{p^2 - 9} \times \frac{1}{p + 2}
= \frac{3}{(p + 2)(p^2 - 9)}.
\]

(c) Find \( \mathcal{L} \left\{ \int_0^x (x - u)^2 \sin 4u \, du \right\} . \)

Here we are taking the Laplace transform with respect to \( x \).

**Method**

Matching the functions in the Convolution theorem, we can see that

\[
f(x - u) = (x - u)^2 \quad \text{and} \quad g(u) = \sin 4u.
\]

Thus,

\[
f(x) = x^2 \quad \text{(replacing} \ x - u \ \text{by} \ x) \quad \text{and} \quad g(x) = \sin 4x
\]

Taking Laplace transforms of both \( f(x) \) and \( g(x) \) with respect to \( x \) by using the Laplace transform tables, we obtain

\[
\mathcal{L} \{ f(x) \} = F(p) = \frac{\Gamma(3)}{p^3} \quad \text{and} \quad \mathcal{L} \{ g(x) \} = G(p) = \frac{4}{p^2 + 16}.
\]

Therefore,

\[
\mathcal{L} \left\{ \int_0^x (x - u)^2 \sin u \, du \right\} = \frac{\Gamma(3)}{p^3} \times \frac{4}{p^2 + 16}
= \frac{4\Gamma(3)}{p^3(p^2 + 16)}
= \frac{8}{p^3(p^2 + 16)}.
\]
5.2.3 Laplace Transform of a Derivative

The Laplace transform of a derivative is:

\[ \mathcal{L}\{f(t)\} = F(p) \]
\[ \mathcal{L}\{f'(t)\} = pF(p) - f(0) \]
\[ \mathcal{L}\{f''(t)\} = p^2 F(p) - pf(0) - f'(0) \]
\[ \mathcal{L}\{f'''(t)\} = p^3 F(p) - p^2 f(0) - pf'(0) - f''(0) \]

These results can be used to solve linear differential equations, and systems of linear differential equations.

Examples

(a) Find the Laplace transform of \( \frac{dy}{dx} - 3y \) where \( y(0) = 1 \).

Method

Using the above information, we let \( \mathcal{L}\{y(x)\} = Y(p) \) then

\[ \mathcal{L}\left\{ \frac{dy}{dx} \right\} = pY(p) - y(0) \]
\[ = pY(p) - 1, \quad \text{where } y(0) = 1. \]

Therefore,

\[ \mathcal{L}\left\{ \frac{dy}{dx} - 3y \right\} = pY(p) - 1 - 3Y(p) \]
\[ = (p - 3)Y(p) - 1 \]

(b) Find \( \mathcal{L}\left\{ \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y \right\} \) where \( y(0) = 2 \) and \( y'(0) = -1 \).

Method

Let \( \mathcal{L}\{y(x)\} = Y(p) \) then

\[ \mathcal{L}\left\{ \frac{dy}{dx} \right\} = pY(p) - y(0) \quad \text{and} \quad \mathcal{L}\left\{ \frac{d^2y}{dx^2} \right\} = p^2 Y(p) - py(0) - y'(0). \]

Using the given conditions we find that

\[ \mathcal{L}\left\{ \frac{dy}{dx} \right\} = pY(p) - 2 \quad \text{and} \quad \mathcal{L}\left\{ \frac{d^2y}{dx^2} \right\} = p^2 Y(p) - 2p + 1. \]

Therefore,

\[ \mathcal{L}\left\{ \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y \right\} = \mathcal{L}\left\{ \frac{d^2y}{dx^2} \right\} - \mathcal{L}\left\{ \frac{dy}{dx} \right\} + \mathcal{L}\{4y\} \]
\[ = p^2 Y(p) - 2p + 1 - (pY(p) - 2) + 4Y(p) \]
\[ = (p^2 - p + 4)Y(p) - 2p + 3. \]
5.2.4 Laplace Transform of Other Types

Recall that
\[ \mathcal{L} \{ tf(t) \} = -\frac{d}{dp} F(p) \quad \text{where} \quad \mathcal{L} \{ f(t) \} = F(p). \]

**Examples**

(a) Find the Laplace transform of \( t \sinh 2t \).

**Method**

Let \( f(t) = \sinh 2t \) then
\[ F(p) = \frac{2}{p^2 - 4}. \]

From Laplace tables where \( n = 2 \).

Therefore,
\[
\mathcal{L} \{ t \sinh 2t \} = -\frac{d}{dp} F(p)
= -\frac{d}{dp} \left( \frac{2}{p^2 - 4} \right)
= \frac{4p}{(p^2 - 4)^2}.
\]

(b) Let \( y = y(t) \). Find the Laplace transform of \( ty'' + y' - ty \) where \( y(0) = 2 \) and \( y'(0) = 0 \).

**Method**

Let \( \mathcal{L} \{ y(t) \} = Y(p) \) then
\[
\mathcal{L} \{ ty \} = -\frac{d}{dp} Y(p) \quad \text{that is,} \quad \mathcal{L} \{ ty \} = -Y'(p).
\]

Also,
\[
\mathcal{L} \{ y' \} = pY(p) - y(0) \quad \text{that is,} \quad \mathcal{L} \{ y' \} = pY(p) - 2.
\]

and
\[
\mathcal{L} \{ ty'' \} = -\frac{d}{dp} \left[ p^2 Y(p) - py(0) - y'(0) \right] \quad \text{that is,} \quad \mathcal{L} \{ ty'' \} = -p^2 Y''(p) - 2pY(p) + 2.
\]

Therefore,
\[
\mathcal{L} \{ ty'' + y' - ty \} = \mathcal{L} \{ ty'' \} + \mathcal{L} \{ y' \} - \mathcal{L} \{ ty \}
= -p^2 Y''(p) - 2pY(p) + 2 + (pY(p) - 2) - (-Y'(p))
= (1 - p^2) Y'(p) - pY(p).
\]

**Note:**

Students should become familiar with Laplace transforms, its properties and the Table of Laplace transforms which are at the end of this chapter, as well as, the inside of the back cover.
**Exercise 5A**

1. This question refers to the table of Basic Laplace Transforms in section 5.6.
   
   (a) Derive each of the results on the first 8 lines of the table by using the definition of a Laplace transform.
   
   (b) From the definition, evaluate the following Laplace transforms.
   
   (i) \( L\{t \sin nt\} \)
   
   (ii) \( L\{t \cos nt\} \)
   
   (iii) \( L\{e^t \sin nt\} \)
   
   (iv) \( L\{e^t \cos nt\} \)
   
   (v) \( L\{\sin nt - nt \cos nt\} \)
   
   (vi) \( L\{e^{3t}(\sin nt + \cos nt)\} \)
   
   (vii) \( L\{t(\sinh 3t + \cos nt)\} \)

2. Find the Laplace transform of the following functions.
   
   (a) \( f(t) = t^2e^{3t} + 1 \)   
   
   (b) \( g(t) = t^2 \sinh 3t \)
   
   (c) \( f(x) = \cos x \sin x \)   
   
   (d) \( g(x) = \cos^2 x \)
   
   (e) \( g(t) = \begin{cases} t & t > 1 \\ 1 & 0 < t < 1 \end{cases} \)
   
   (f) \( f(x) = \begin{cases} \cos x & x > \frac{\pi}{2} \\ \sin x & 0 < x < \frac{\pi}{2} \end{cases} \)
   
   (g) \( k(t) = \int_0^t e^{-u} \sin u \, du \)
   
   (h) \( w(t) = \int_0^t \sin 2(t - u) \sin u \, du \)
   
   (i) \( k(t) = \int_0^t \cos(t - u) \sin 3u \, du \)

3. (a) Verify the following results.
   
   (i) \( \int_0^t f(t - u) \, du = \int_0^t f(u) \, du \)
   
   (ii) \( \int_0^t f(t - u) g(u) \, du = \int_0^t f(u) g(t - u) \, du \)

4. Use the table of Laplace transforms to write down the values of the following integrals.
   
   (a) \( \int_0^\infty e^{-5t} \, dt \)
   
   (b) \( \int_0^\infty e^{-2t^2} \, dt \)
   
   (c) \( \int_0^\infty e^{-2t} \cos 5t \, dt \)
   
   (d) \( \int_0^\infty e^{-4t} \cosh 3t \, dt \)
   
   (e) \( \int_0^\infty e^{-3t} \int_0^t e^{-u} \sin u \, du \, dt \)
   
   (f) \( \int_0^\infty e^{-x} \int_0^x \sin 2(x - u) \cos u \, du \, dx \)

5. (a) Show that \( L\left\{\frac{1}{x} f(x)\right\} = \int_p^\infty F(p) \, dp \).
   
   (b) Hence determine the Laplace transforms of the following functions:
   
   (i) \( \frac{\sin ax}{x} \)
   
   (ii) \( \frac{1 - \cos ax}{x} \)
   
   (iii) \( \frac{e^{ax} - e^{bx}}{x} \)

6. (a) Use the definition of the Laplace Transform, and integration by parts, to find the Laplace transforms of \( y(t) \), \( y''(t) \).
   
   (b) Let \( L\{y(t)\} = Y(p) \). By taking Laplace transforms, find \( Y(p) \) for each of the following equations subject to the given initial conditions.
   
   (i) \( \frac{dy}{dt} - y = e^{-t} \) where \( y(0) = 1 \).
   
   (ii) \( 2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 5y = \sin 2t \)
   
   where \( y(0) = -1 \) and \( y'(0) = 1 \).
5.3 LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS

In this section we will look at the Laplace transform of some important special functions.

5.3.1 Step (Heaviside) Function

Recall the step function definition:

\[ h(t - a) = \begin{cases} 
1 & t \geq a \\
0 & t < a. 
\end{cases} \]

Then the Laplace transform of \( h(t - a) \) is

\[
\mathcal{L}\{h(t-a)\} = \int_{0}^{\infty} h(t-a)e^{-pt} \, dx \\
= \int_{0}^{\infty} e^{-pt} \, dx \\
= \frac{e^{-ap}}{p}.
\]

Therefore, the inverse Laplace of \( \frac{e^{-ap}}{p} \) is \( h(t - a) \). That is,

\[
\mathcal{L}^{-1}\left\{ \frac{e^{-ap}}{p} \right\} = h(t-a). 
\]

Property

\[
\mathcal{L}\{f(t-a)h(t-a)\} = e^{-ap}F(p)
\]

where \( \mathcal{L}\{f(t)\} = F(p) \).

Examples

(a) Find \( \mathcal{L}\{h(t-3)\} \).

Method

Recall that \( \mathcal{L}\{h(t-a)\} = \frac{e^{-ap}}{p} \) then

\[
\mathcal{L}\{h(t-3)\} = \frac{e^{-3p}}{p}, \quad \text{where} \quad a = 3. 
\]

(b) Find \( \mathcal{L}\{th(t-2)\} \).

Method

Reshaping the given function we have

\[ th(t-2) = (t-2)h(t-2) + 2h(t-2). \]

Therefore,

\[
\mathcal{L}\{th(t-2)\} = \mathcal{L}\{(t-2)h(t-2)\} + 2\mathcal{L}\{h(t-2)\}. 
\]
Consider the first term on the right hand side of the equation. Noting the property that
\[
\mathcal{L}\{ f(t-a)h(t-a) \} = e^{-ap} F(p) \quad \text{where} \quad F(p) = \mathcal{L}\{ f(t) \}
\]
then
\[
f(t-2) = t - 2 \quad \text{where} \quad a = 2.
\]
Hence,
\[
f(t) = t \quad \text{where} \quad t - 2 \text{ is replaced by } t.
\]
Therefore,
\[
\mathcal{L}\{ f(t) \} = F(p) = \mathcal{L}\{ t \}
= \frac{\Gamma(2)}{p^2}
= \frac{1}{p^2}.
\]
\[
\Rightarrow \quad \mathcal{L}\{ (t-2)h(t-2) \} = \frac{e^{-2p}}{p^2}, \quad \text{where} \quad a = 2.
\]
Also,
\[
\mathcal{L}\{ 2h(t-a) \} = \frac{2e^{-2p}}{p} \quad \text{where} \quad a = 2.
\]
As a result we find that
\[
\mathcal{L}\{ t h(t-2) \} = \frac{e^{-2p}}{p^2} + \frac{2e^{-2p}}{p} = \frac{e^{-2p}}{p^2}(1 + 2p).
\]

### 5.3.2 Dirac Delta (Impulse) Function

Recall that
\[
f_k(t-a) = \begin{cases} \frac{1}{k} & a < t < a + k \\ 0 & \text{otherwise.} \end{cases}
\]
The limit of \( f_k(t) \) as \( k \to 0 \) is denoted by \( \delta(t-a) \). Now
\[
\mathcal{L}\{ f_k(t-a) \} = \int_0^\infty e^{-pt} f_k(t-a) \, dt
= \int_a^{a+k} e^{-pt} \frac{1}{k} \, dt
= \frac{1}{k} \left[ e^{-pt} \right]_a^{a+k}
= e^{-sa} \left( 1 - e^{-pk} \right) \frac{k}{kp}.
\]
Now
\[
\lim_{k \to 0} \mathcal{L}\{ f_k(t-a) \} = \lim_{k \to 0} e^{-pa} \left( 1 - e^{-pk} \right) \frac{k}{ks} \quad \left( = \frac{0}{0} \right)
\]
\[\text{via}
\lim_{k \to 0} e^{-pa} \left( pe^{-pk} \right) \quad \left( = \frac{0}{0} \right)
\]
\[= e^{-pa}.
\]
Hence,
\[ \mathcal{L}\{\delta(t-a)\} = e^{-pa}. \]

**Example**
Find \( \mathcal{L}\{\delta(x-2)\} \).

**Method**
\[ \mathcal{L}\{\delta(x-2)\} = e^{-2p} \quad \text{From tables, where } a=2. \]

**Note:**
The Dirac Delta function is sometimes called the unit impulse function. It is not a function in the ordinary sense. The definition of the Dirac Delta function can vary depending on the text.

### 5.3.3 Error Function

Recall the definition of the error function. That is,
\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} \, du \]
then \( \mathcal{L}\{\text{erf}(t)\} \) is obtained by taking the Laplace transform with respect to \( t \). That is,
\[
\begin{align*}
\mathcal{L}\{\text{erf}(t)\} &= \int_0^\infty e^{-pt} \text{erf}(t) \, dt \\
&= \int_0^\infty e^{-pt} \left( \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} \, du \right) \, dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left( \int_u^\infty e^{-pt} \, dt \right) \, du \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left( \frac{e^{-pt}}{-p} \right) \, du \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left( \frac{e^{-pt}}{-p} \right) \, du \\
&= \frac{2}{\sqrt{\pi}} e^{-p^2/4} \int_0^\infty e^{-u^2} \, du \\
&= \frac{2}{\sqrt{\pi}} e^{-p^2/4} \left( \frac{1}{2} \right) \int_0^\infty e^{-u^2} \, du \\
&= \frac{e^{-p^2/4}}{2}. \\
\end{align*}
\]
That is,
\[ \mathcal{L}\{\text{erf}(t)\} = \frac{e^{-p^2/4}}{2}. \]

Complete the square.
Let \( z = u + \frac{p}{2} \) then \( dz = du \)
when \( u = 0, z = \frac{p}{2} \); when \( u = \infty, z = \infty \)
\[ e^{-u^2} \text{erfc}(\frac{u}{2}) = \frac{e^{-p^2/4}}{p}. \]
**Exercise 5B**

1. Define the functions $h(t)$, $h(t-a)$, $\delta(t-a)$ and find their Laplace transforms.

   Use the Convolution theorem to show that
   
   $$\mathcal{L}\left\{ \int_0^t h(t-u)x(u) \, du \right\} = \frac{X(p)}{p}. $$

2. Find the Laplace transform of the following functions.
   
   (a) $h(t-2)$
   
   (b) $h(t-2)e^{(t-2)}$
   
   (c) $h(t-2)e^t$
   
   (d) $\sin(3t)\left(h(t-\frac{\pi}{2}) - h(t)\right)$
   
   (e) $\delta(t-5)$
   
   (f) $\text{erfc}(\sqrt{t})$
   
   (g) $\int_0^t h(t-u)\sin 3u \, du$
   
   (h) $\int_0^t h(t-u)\cos 2u \, du$
   
   (i) $e^{-t^2}$
   
   (j) $\int_0^\infty \delta(u-t)\sin 3u \, du$ for $t > 0$
   
   (k) $\int_0^\infty \delta(u-t)\cos 2u \, du$ for $t > 0$
   
   (l) $\int_0^t h(t-u-2)\sin 3u \, du$
   
   (m) $\int_0^t h(t-u-3)\cos 2u \, du$

3. Let $\mathcal{L}\{y(t)\} = Y(p)$. Find $Y(p)$ if
   
   (a) $\frac{dy}{dx} - y = \delta(x-1)$
   
   (b) $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = \delta(t-3)$

4. Use transform definitions, and the evaluation of a suitable double integral, to calculate the Laplace transform of the following integrals.
   
   (i) $\mathcal{L}\left\{ \int_0^t x(u) \, du \right\}$
   
   (ii) $\mathcal{L}\left\{ \int_0^t f(t-u)g(u) \, du \right\}$ (Convolution integral).

5. Evaluate $\int_0^x \text{erf}(\sqrt{x-t}) \text{erf}(\sqrt{t}) \, dt$.

6. Prove the following results.
   
   (a) $\mathcal{L}\{\delta(x)\} = 1$
   
   (b) $\mathcal{L}\{\delta(x-a)\} = e^{-ap}$, $a \geq 0$

7. (a) Show that the Laplace transform of

   $$f(x-a)h(x-a)$$

   is $e^{-ap}F(p)$.

   (b) Hence find the functions whose transforms are given below.

   (i) $\frac{e^{-ap}}{p^2}$, $a > 0$

   (ii) $\frac{e^{-ap}}{p^2 + 1}$.

---

5.4 **Inverse Laplace Transform of a Function**

The inversion formula for the Laplace transform is

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} F(p) \, dp$$

where $\gamma$ is called the *Bromwich Line*.
We usually denote the inverse Laplace transform by $L^{-1}$. That is,

$$f(t) = L^{-1}\{F(p)\}$$

where $f(t)$ is the original function.

*Note:*

The assumption here is that the independent variable of the original function $f$ is $t$ but this not necessary the case.

In this work, we will rely on using the table in section 5.6 to invert many transforms as the inversion formula requires complex analysis methods. However, we will use the Laplace transform properties, the table of Laplace transforms and the Convolution theorem to aid in determining the inverse Laplace transform of a function.

### 5.4.1 Properties of the Inverse Laplace Transform

Let $f(t) = L^{-1}\{F(p)\}$ and $g(t) = L^{-1}\{G(p)\}$ and let $\alpha$ and $\beta$ be two arbitrary constants then

$$L\mathcal{L}^{-1} = L^{-1}L = I$$

where $I$ is the identity operator.

$$L^{-1}\{\alpha F(p)\} = \alpha L^{-1}\{F(p)\}.\$$

$$L^{-1}\{\alpha F(p) + \beta G(p)\} = \alpha L^{-1}\{F(p)\} + \beta L^{-1}\{G(p)\}\$$

$$L^{-1}\{F(p + a)\} = e^{-at}f(t)$$

(Shift Theorem)

### 5.4.2 Techniques for Finding the Inverse Laplace Transform

There are various techniques to finding the inverse Laplace transform of a function. We shall specifically look at four techniques.

**Technique 1: Standard Tables**

**Examples**

(a) Find the inverse Laplace transform of $F(p) = \frac{1}{p^2}$.

**Method**

From the tables it can be seen that

$$\mathcal{L}\{t^r\} = \frac{\Gamma(r + 1)}{p^{r+1}} \quad \text{for} \quad r > -1.$$

Let $L^{-1}\{F(p)\} = f(t)$ then

$$f(t) = L^{-1}\left\{\frac{1}{p^{r+1}}\right\} = \frac{t^r}{\Gamma(r+1)}.$$
In comparison to our example, it can be seen that $r + 1 = 2$. Therefore, $r = 1$ and

\[ f(t) = L^{-1}\left\{\frac{1}{p^2}\right\} = \frac{t}{\Gamma(2)} = t. \]

(b) Find the inverse Laplace transform of $\frac{1}{p^2 + 4}$.

Method

From the tables it can be seen that

\[ L\{\sin 2t\} = \frac{2}{p^2 + 4}, \quad \text{where} \quad n = 2. \]

Therefore,

\[ L^{-1}\left\{\frac{1}{p^2 + 4}\right\} = \frac{1}{2} L^{-1}\left\{\frac{2}{p^2 + 4}\right\} = \frac{1}{2} \sin 2t. \]

(c) Find the inverse Laplace transform of $\frac{1}{p - 3}$.

Method

From the tables it can be seen that

\[ L\{e^{3t}\} = \frac{1}{p - 3}, \quad \text{where} \quad b = -3. \]

then

\[ L^{-1}\left\{\frac{1}{p - 3}\right\} = e^{3t}. \]

Note:

The assumption in these examples is that $L^{-1}\{F(p)\} = f(t)$. That is, the original function $f$ is a function of the independent variable $t$ but this is not necessarily the case. We can assume any independent variable if our problem does not specify.

Technique 2: Rational Functions

If $F(p)$ is a rational function of $p$ then we can use partial fractions to simplify the rational function and tables to determine $L^{-1}\{F(p)\}$.

Examples

(a) Find the inverse Laplace transform of $\frac{1}{p(p - 1)}$.

Method

We can see that

\[ \frac{1}{p(p - 1)} = A \frac{1}{p} + B \frac{1}{p - 1} \]

where $A$ and $B$ need to be determined by the usual partial fraction technique. It is easily shown that $A = -1$ and $B = 1$. Therefore,
\[
L^{-1}\left\{\frac{1}{p(p-1)}\right\} = L^{-1}\left\{\frac{-1}{p} + \frac{1}{p-1}\right\} \\
= L^{-1}\left\{-\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p-1}\right\} \\
= -1 + e^t, \quad \text{using tables.}
\]

(b) Find \(L^{-1}\left\{\frac{6}{p^2(p^2 + 9)}\right\} \).

Method

Using partial fractions we have \(\frac{6}{p^2(p^2 + 9)} = \frac{A}{p} + \frac{B}{p^2} + \frac{Cp + D}{p^2 + 9}\) where \(A\), \(B\), \(C\) and \(D\) need to be determined by the usual partial fraction method. It can be shown that \(A = 0\), \(B = \frac{2}{3}\), \(C = 0\) and \(D = -\frac{2}{3}\). Therefore,

\[
L^{-1}\left\{\frac{6}{p^2(p^2 + 9)}\right\} = L^{-1}\left\{\frac{\frac{2}{3}}{p^2} - \frac{\frac{2}{3}}{p^2 + 9}\right\} \\
= \frac{2}{3} L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{2}{3} L^{-1}\left\{\frac{1}{p^2 + 9}\right\} \\
= \frac{2}{3} \Gamma(2) - \frac{2}{9} L^{-1}\left\{\frac{3}{p^2 + 9}\right\} \quad \text{using tables} \\
= \frac{2}{3} t - \frac{2}{9} \sin 3t, \quad \text{using tables.}
\]

Technique 3: Shift Theorem

(a) Find the inverse Laplace transform of \(\frac{1}{(p + 1)^2}\).

Method

Recall the Shift theorem. That is,

\[L\{e^{-at}f(t)\} = F(p + a) \quad \text{or} \quad e^{-at}f(t) = L^{-1}\{F(p + a)\}.
\]

That is,

\[L^{-1}\{F(p + a)\} = e^{-at}L^{-1}\{F(p)\}.
\]

In our example we let

\[F(p + 1) = \frac{1}{(p + 1)^2}, \quad \text{where} \quad a = 1.
\]

Therefore,

\[F(p) = \frac{1}{p^2}.
\]

Now

\[
L^{-1}\{F(p)\} = L^{-1}\left\{\frac{1}{p^2}\right\} = \frac{t}{\Gamma(2)}. \quad \text{From tables where} \quad r + 1 = 2.
\]
Hence,
\[
\mathcal{L}^{-1}\left\{\frac{1}{(p+1)^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} = e^{-t}\frac{t}{\Gamma(2)} = te^{-t}.
\]

(b) Find the inverse Laplace transform of \(\frac{1}{(p-3)^2 + 4}\).

Method
Here we let
\[
F(p-3) = \frac{1}{(p-3)^2 + 4}, \quad \text{where} \quad a = -3.
\]
then
\[
F(p) = \frac{1}{p^2 + 4}.
\]
Recall that
\[
\mathcal{L}\{\sin 2t\} = \frac{2}{p^2 + 4}, \quad \text{where} \quad n = 2.
\]
then
\[
\mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{p^2 + 4}\right\} = \frac{1}{2}\sin 2t.
\]
Therefore,
\[
\mathcal{L}^{-1}\left\{\frac{1}{(p-3)^2 + 4}\right\} = e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{p^2 + 4}\right\} = e^{3t}\times\frac{1}{2}\sin 2t = \frac{e^{3t}}{2}\sin 2t.
\]

(c) Find the inverse Laplace transform of \(\frac{1}{p^2 + 4p + 1}\).

Method
In this example we will have to complete the square before being able to use the shift theorem. That is,
\[
\frac{1}{p^2 + 4p + 1} = \frac{1}{(p + 2)^2 - 3}.
\]
Therefore, let
\[
F(p + 2) = \frac{1}{(p + 2)^2 - 3} \quad \text{then} \quad F(p) = \frac{1}{p^2 - 3}.
\]
Form the Laplace transform tables we see that
\[
\mathcal{L}^{-1}\left\{\frac{1}{p^2 - 3}\right\} = \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{p^2 - 3}\right\} = \frac{1}{\sqrt{3}}\sinh(\sqrt{3}t) \quad \text{where} \quad n = \sqrt{3}.
\]
Therefore,
\[
\mathcal{L}^{-1} \left\{ \frac{1}{(p+2)^2 - 3} \right\} = e^{-2t} \frac{1}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{e^{-2t}}{\sqrt{3}} \sinh(\sqrt{3} t).
\]

(d) Find the inverse Laplace transform of \( \frac{p}{p^2 + 4p + 1} \).

\textbf{Method}

Once again we complete the square in the denominator before using the shift theorem. That is,
\[
\frac{p}{p^2 + 4p + 1} = \frac{p}{(p+2)^2 - 3}.
\]

The right hand side is not in the form of \( F(p+a) \) as yet. This is due to the numerator being only a function of \( p \). Therefore, we manipulate the rational function so that we do obtain the this form. That is,
\[
\frac{p}{(p+2)^2 - 3} = \frac{p+2}{(p+2)^2 - 3} - \frac{2}{(p+2)^2 - 3}
= F_1(p+2) - F_2(p+2).
\]

Now
\[
F_1(p+2) = \frac{p+2}{(p+2)^2 - 3} \quad \text{then} \quad F_1(p) = \frac{p}{p^2 - 3}.
\]

From the Laplace transform tables we see that
\[
\mathcal{L}^{-1} \left\{ F_1(p) \right\} = \mathcal{L}^{-1} \left\{ \frac{p}{p^2 - 3} \right\} = \cosh \sqrt{3} t.
\]

Therefore,
\[
\mathcal{L}^{-1} \left\{ \frac{p+2}{(p+2)^2 - 3} \right\} = e^{-2t} \cosh \sqrt{3} t.
\]

Consider \( F_2(p+2) \). That is,
\[
F_2(p+2) = \frac{2}{(p+2)^2 - 3} \quad \text{then} \quad F_2(p) = \frac{2}{p^2 - 3}.
\]

From the Laplace transform tables we see that
\[
\mathcal{L}^{-1} \left\{ F_2(p) \right\} = \mathcal{L}^{-1} \left\{ \frac{2}{p^2 - 3} \right\} = \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{2}{\sqrt{3}} \sinh \sqrt{3} t.
\]
Therefore,
\[
\mathcal{L}^{-1} \left\{ \frac{1}{(p+2)^2 - 3} \right\} = e^{-2t} \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{2e^{-2t}}{\sqrt{3}} \sinh \sqrt{3} t.
\]

Hence,
\[
\mathcal{L}^{-1} \left\{ \frac{p}{p^2+4p+1} \right\} = e^{-2t} \cosh \sqrt{3} t - \frac{2e^{-2t}}{\sqrt{3}} \sinh \sqrt{3} t.
\]

(c) Find \( \mathcal{L}^{-1} \left\{ \frac{1}{p(p^2+4p+1)} \right\} \).

Method

Using partial fractions it is found that
\[
\frac{1}{p(p^2+4p+1)} = \frac{1}{p} - \frac{p+4}{p^2+4p+1}.
\]

Therefore,
\[
\mathcal{L}^{-1} \left\{ \frac{1}{p(p^2+4p+1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{p} - \frac{p+4}{p^2+4p+1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{p} \right\} - \mathcal{L}^{-1} \left\{ \frac{p+4}{p^2+4p+1} \right\}.
\]

Now
\[
\mathcal{L}^{-1} \left\{ \frac{1}{p} \right\} = 1.
\]

The second term requires to be put into a form so that we can use one of our inverse Laplace transform techniques. This can be done by completing the square and adding and subtracting constants. That is,
\[
\frac{p+4}{p^2+4p+1} = \frac{p+4}{(p+2)^2 - 3} = \frac{p+4}{(p+2)^2 - 3} - \frac{2}{(p+2)^2 - 3}.
\]

Consider the first term on the right hand side and let \( F(p+2) = \frac{p+2}{(p+2)^2 - 3} \) then \( F(p) = \frac{p}{p^2 - 3} \).

Therefore,
\[
\mathcal{L}^{-1} \{ F(p) \} = \mathcal{L}^{-1} \left\{ \frac{p}{p^2 - 3} \right\} = \cosh(\sqrt{3}t) \text{ using tables where } n = \sqrt{3}.
\]

Consider the second term on the right hand side and let \( F(p+2) = \frac{2}{(p+2)^2 - 3} \) then \( F(p) = \frac{2}{p^2 - 3} \).

Therefore,
\[
\mathcal{L}^{-1} \{ F(p) \} = \mathcal{L}^{-1} \left\{ \frac{2}{p^2 - 3} \right\} = \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{2}{\sqrt{3}} \sinh(\sqrt{3}t) \text{ using tables where } n = \sqrt{3}.
\]

Therefore,
\[
\mathcal{L}^{-1} \left\{ \frac{1}{p(p^2+4p+1)} \right\} = 1 + \left( \cosh(\sqrt{3}t) + \frac{2}{\sqrt{3}} \sinh(\sqrt{3}t) \right) e^{-2t}.
\]
Technique 4: Convolution Theorem

Taking the Laplace transform of a product of two functions may be easily obtained by using the convolution theorem rather than the previous techniques. This will depend on the given function. We shall solve some of the previous examples using this technique.

Recall that
\[
\mathcal{L}\left\{ \int_0^t f(t-u)g(u)\,du \right\} = F(p)G(p)
\]
or that
\[
\mathcal{L}\left\{ \int_0^t f(u)g(t-u)\,du \right\} = F(p)G(p)
\]
then
\[
\mathcal{L}^{-1}\{F(p)G(p)\} = \int_0^t f(t-u)g(u)\,du.
\]

Examples

(a) Find \( \mathcal{L}^{-1}\left\{ \frac{1}{p(p-1)} \right\} \).

Method

We can see that
\[
\frac{1}{p(p-1)} = \frac{1}{p} \times \frac{1}{p-1}.
\]

Therefore, let
\[
F(p) = \frac{1}{p} \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{ \frac{1}{p} \right\} = 1.
\]

and
\[
G(p) = \frac{1}{p-1} \quad \Rightarrow \quad g(t) = \mathcal{L}^{-1}\{G(p)\} = \mathcal{L}^{-1}\left\{ \frac{1}{p-1} \right\} = e^t.
\]

Now
\[ f(t-u) = 1, \quad \text{Replacing } t \text{ by } t-u. \]

and
\[ g(u) = e^u, \quad \text{Replacing } t \text{ by } u. \]

Using the Convolution theorem we have
\[
\mathcal{L}^{-1}\left\{ \frac{1}{p(p-1)} \right\} = \int_0^t 1 \cdot e^u\,du \quad \text{substituting for } f(t-u) \text{ and } g(u)
\]
into the Convolution theorem.
\[
= e^t\bigg|_0^t
\]
\[
= e^t - 1.
\]

This is the same answer as that obtained in Example (a) using Technique two.

(b) Find \( \mathcal{L}^{-1}\left\{ \frac{6}{p^2(p^2 + 9)} \right\} \).

Method

We can see that
\[
\frac{6}{p^2(p^2 + 9)} = \frac{2}{p^2} \times \frac{3}{p^2 + 9}.
\]
Therefore, let
\[ F(p) = \frac{2}{p^2} \implies f(t) = \mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{ \frac{2}{p^2} \right\} = 2t \]
and
\[ G(p) = \frac{3}{p^2 + 9} \implies g(t) = \mathcal{L}^{-1}\{G(p)\} = \mathcal{L}^{-1}\left\{ \frac{3}{p^2 + 9} \right\} = \sin 3t. \]

Now
\[ f(t - u) = 2(t - u), \quad \text{Replacing } t \text{ by } t - u. \]
and
\[ g(u) = \sin 3u, \quad \text{Replacing } t \text{ by } u. \]

Using the Convolution theorem we have that
\[
\mathcal{L}^{-1}\left\{ \frac{6}{p^2(p^2 + 9)} \right\} = \int_0^t 2(t - u) \sin 3u \, du \quad \text{substituting for } f(t - u) \text{ and } g(u) \\
\text{into the Convolution theorem.}
\]
\[
= 2 \int_0^t (t - u) \sin 3u \, du, \quad \text{Integrate by parts.}
\]
\[
= 2 \left\{ \left( t - u \right) \frac{\cos 3u}{-3} \right|_0^t - \int_0^t (-1) \frac{\cos 3u}{-3} \, du \right\}
\]
\[
= 2 \left\{ \frac{t}{3} + \frac{\sin 3u}{9} \right|_0^t \right\}
\]
\[
= 2 \left\{ \frac{2}{3} t - \frac{2 \sin 3t}{9} \right\}.
\]
This is the same as that obtained in Example (b) using Technique two.

### 5.5 INVERSE LAPLACE TRANSFORM OF SPECIAL FUNCTIONS

**Examples**

(a) Find \( \mathcal{L}^{-1}\left\{ \frac{e^{-3p}}{p} \right\} \).

**Method**

Using Laplace transform tables, we find that

\[
\mathcal{L}^{-1}\left\{ \frac{e^{-3p}}{p} \right\} = h(t - 3), \quad \text{where } a = 3.
\]

(b) Find \( \mathcal{L}^{-1}\left\{ \frac{e^{-4p}}{p^2} \right\} \).

**Method**

Recall that

\[
\mathcal{L}\{f(t-a)h(t-a)\} = e^{-ap}F(p) \implies \mathcal{L}^{-1}\left\{ e^{-ap}F(p) \right\} = f(t-a)h(t-a),
\]

where \( \mathcal{L}\{f(t)\} = F(p) \).
Now
\[ \frac{e^{-4p}}{p^2} = e^{-4p} \times \frac{1}{p^2}. \]
Therefore, let \( F(p) = \frac{1}{p^2} \) and \( a = 4 \).

Hence,
\[ f(t) = \mathcal{L}^{-1}\{F(p)\} = \frac{t}{1(2)} = t \quad \implies \quad f(t - 4) = t - 4. \]

So that
\[ \mathcal{L}^{-1}\left\{\frac{e^{-4p}}{p^2}\right\} = (t - 4)h(t - 4). \]

(c) Find \( \mathcal{L}^{-1}\left\{\frac{pe^{-p}}{p^2 + 1}\right\}. \)

\textbf{Method}
\[ \frac{pe^{-p}}{p^2 + 1} = e^{-p} \times \frac{p}{p^2 + 1}. \]
Therefore, let \( F(p) = \frac{p}{p^2 + 1} \) and \( a = 1 \).

Hence,
\[ f(t) = \mathcal{L}^{-1}\{F(p)\} = \cos t \quad \implies \quad f(t - 1) = \cos(t - 1). \]

So that
\[ \mathcal{L}^{-1}\left\{\frac{pe^{-p}}{p^2 + 1}\right\} = \cos(t - 1)h(t - 1). \]

(d) Find \( \mathcal{L}^{-1}\{e^{-2p}\}. \)

\textbf{Method}
Refer to the Laplace transform tables
\[ \mathcal{L}^{-1}\{e^{-2p}\} = \delta(t - 2) \quad \text{where} \quad a = 2. \]

(e) Find \( \mathcal{L}^{-1}\left\{\frac{e^{-2p}}{(p + 1)^2}\right\}. \)

\textbf{Method}
\[ \frac{e^{-2p}}{(p + 1)^2} = e^{-2p} \times \frac{1}{(p + 1)^2}. \]
Therefore, let \( F(p) = \frac{1}{(p + 1)^2} \) and \( a = 2 \).

Now
\[ \mathcal{L}^{-1}\left\{\frac{1}{(p + 1)^2}\right\} = e^{-t} \mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} \quad \text{where} \quad r + 1 = 2 \quad \text{and} \quad b = 1 \]
\[ = e^{-t} \frac{t}{\Gamma(2)}, \quad \text{using tables.} \]
\[ = te^{-t}. \]

Therefore,
\[ f(t) = \mathcal{L}^{-1}\{F(p)\} = te^{-t} \quad \implies \quad f(t - 2) = (t - 2)e^{-(t - 2)}. \]

So that
\[ \mathcal{L}^{-1}\left\{\frac{e^{-2p}}{(p + 1)^2}\right\} = (t - 2)e^{-(t - 2)}h(t - 2). \]
Exercise 5C

1 Use partial fractions or the convolution theorem and the Table of Laplace Transforms, to find functions of \( t \) which have the following Laplace transforms.

\[
\begin{align*}
(a) & \quad \frac{2}{3p+2} \\
(b) & \quad \frac{p+2}{2p^2+p-1} \\
(c) & \quad \frac{1}{p(p^2+4)} \\
(d) & \quad \frac{p^2}{(p+3)^3} \\
(e) & \quad \frac{1}{p^2(p+1)(p^2+4)} \\
(f) & \quad \frac{7p-13}{p^2(p+6p+13)} \\
(g) & \quad \frac{1}{p^2(p^2-4)} \\
(h) & \quad \frac{1}{p^2(p-1)^2} \\
(i) & \quad \frac{2p}{p^2+2p+5} \\
(j) & \quad \frac{p-2}{p^2(p^2+4)}
\end{align*}
\]

2 (a) Show that the Laplace transform of \( f(x-a)h(x-a) \) is \( e^{-ap}F(p) \).

(b) Hence find the functions whose transforms are given below.

\[
\begin{align*}
(i) & \quad \frac{e^{-ap}}{p^2}, \quad a \geq 0 \\
(ii) & \quad \frac{e^{-xp}}{p^2+1} \\
(iii) & \quad \frac{e^{-4p}}{p^2+2p+5} \\
(iv) & \quad \frac{e^{-3p}}{p(p+2)}
\end{align*}
\]

5.6 INTEGRAL AND DIFFERENTIAL EQUATIONS

Laplace transforms can be used to solve both integral and linear differential equations. Consider the following initial value problem (IVP)

\[ ay'' + by'' + cy = g(t) \]

subject to \( y(0) = \alpha \) and \( y'(0) = \beta \).

Here the given differential equation is a constant coefficient non-homogeneous differential equation with given initial conditions. That is, \( a, \ b \) and \( c \) are constants.

The procedure is to transform the given differential equation to an algebraic equation which in turn may be easily solved. This procedure is shown diagramatically below.

Taking transforms of both sides of the given differential equation we have

\[ a(p^2Y(p) - p y(0) - y'(0)) + b(pY(p) - y(0)) + cY(p) = G(p). \]
Applying the given initial conditions and solving for $Y(p)$, it is found that

$$Y(p) = \frac{1}{a^2p^2 + bp + c} (\alpha + \beta + G(p)).$$

This is can be represented in the form

$$Y(p) = \frac{1}{P(p)} (Q(p) + G(p))$$

where

$$P(p) = a^2p^2 + bp + c, \quad Q(p) = \alpha + \beta \quad \text{and} \quad G(p) = \mathcal{L} \{ g(t) \}.$$  

In this manner we have separated the effects of the initial conditions and those due to the input function $g(t)$.

The reciprocal of $P(p)$, that is, $\frac{1}{P(p)}$ is called the transfer function of the given equation (system).

$P(p) = 0$ is the auxiliary/characteristic equation of the differential equation using normal analytic methods.

Upon taking inverse Laplace transforms we have that

$$y(t) = \mathcal{L}^{-1} \{ \frac{Q(p)}{P(p)} \} + \mathcal{L}^{-1} \{ \frac{G(p)}{P(p)} \}.$$  

If $g(t) = 0$ then the solution $\mathcal{L}^{-1} \{ \frac{Q(p)}{P(p)} \}$ of the problem is called the zero input response and when the initial conditions are all zero, the solution $\mathcal{L}^{-1} \{ \frac{G(p)}{P(p)} \}$ is called the zero state response.

Examples

(a) Solve the following differential equation

$$y'' - y' - 2y = e^t$$

subject to $y(0) = 0$ and $y'(0) = 0$.

Method

Take Laplace transforms of the differential equation with respect to $t$. That is,

$$\mathcal{L} \{ y'' - y' - 2y = e^t \}$$

gives

$$p^2Y(p) - py(0) - y'(0) - (pY(p) - y(0)) - 2Y(p) = \frac{1}{p - 1}.$$  

Using the given initial conditions and simplifying, we have

$$Y(p) = \frac{1}{(p - 1)(p^2 - p - 2)} = \frac{1}{(p^2 - 1)(p - 2)}.$$  

At this stage we can use partial fractions or the Convolution theorem. In this, example we shall use the Convolution theorem.
Let \( F(p) = \frac{1}{(p - 2)} \) and \( G(p) = \frac{1}{(p^2 - 1)} \).

Note: The choice of \( F(p) \) and \( G(p) \) is arbitrary.

Now
\[
\mathcal{L}^{-1}\{F(p)G(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{(p - 2)}\right\} \times \mathcal{L}^{-1}\left\{\frac{1}{(p^2 - 1)}\right\}
\]
then
\[
f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(p - 2)}\right\} = e^{2t}
\]
and
\[
g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(p^2 - 1)}\right\} = \sinh t.
\]

Therefore,
\[
f(t - u) = e^{2(t - u)} \quad \text{and} \quad g(u) = \sinh u.
\]

Using the Convolution theorem we find that the solution to the differential equation is
\[
y(t) = L^{-1}\left\{\frac{1}{(p - 1)(p^2 - p - 2)}\right\} = \int_0^t e^{2(t - u)} \sinh u \, du
\]
\[
= -\frac{e^t}{2} + \frac{e^{2t}}{3} + \frac{e^{-t}}{6}.
\]

Our solution can be checked by using \( D \) operator techniques. We find that the general solution to the differential equation is
\[
y(t) = Ae^{-t} + Be^{2t} + \frac{e^t}{2}.
\]

Using the initial conditions we find that
\[
y = \frac{1}{6}e^{-t} + \frac{1}{3}e^{2t} - \frac{e^t}{2}.
\]

(b) Solve, using Laplace transforms, the following differential equation
\[
\frac{dy}{dx} + 2y = \cos 2x,
\]
subject to \( y(0) = 1 \).

Method

Take Laplace transforms of the differential equation with respect to \( x \). That is,
\[
\mathcal{L}\left\{\frac{dy}{dx} + 2y = \cos 2x\right\}
\]
gives
\[
pY(p) - y(0) + 2Y(p) = \frac{p}{p^2 + 4}.
\]

Using the given initial conditions and simplifying, we have
\[
Y(p) = \frac{1}{p + 2} + \frac{p}{(p + 2)(p^2 + 4)}.
\]
Now
\[ \mathcal{L}^{-1} \left\{ \frac{1}{p + 2} \right\} = e^{-2x} \]
and
\[ \mathcal{L}^{-1} \left\{ \frac{p}{(p + 2)(p^2 + 4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(p + 2)} \times \frac{p}{(p^2 + 4)} \right\} . \]

Let \( F(p) = \frac{1}{p + 2} \) and \( G(p) = \frac{p}{p^2 + 4} \).

Using the Laplace transform tables we find that 
\[ \mathcal{L}^{-1} \left\{ \frac{1}{(p + 2)} \right\} = e^{-2x} \] and 
\[ \mathcal{L}^{-1} \left\{ \frac{p}{(p^2 + 4)} \right\} = \cos 2x . \]

Then
\[ f(x) = e^{-2x} \quad g(x) = \cos 2x. \]

Hence,
\[ f(x - u) = e^{-2(x-u)} \quad g(u) = \cos 2u. \]

Therefore, the Convolution theorem produces
\[ \mathcal{L}^{-1} \left\{ \frac{p}{(p + 2)(p^2 + 4)} \right\} = \int_0^x e^{-2(x-u)} \cos 2u \, du \]
\[ = e^{-2x} \int_0^x e^{2u} \cos 2u \, du. \]

The integral on the right hand side of the above equation is a recurring integral after integrating by parts. The working is as follows.

Let
\[ I = e^{-2x} \int_0^x e^{2u} \cos 2u \, du \]

integrate by parts
\[ = e^{-2x} \left\{ \frac{e^{2u} \cos 2u}{2} \right\} \bigg|_0^x + \int_0^x e^{2u} \sin 2u \, du \right\} \]
integrate by parts
\[ = e^{-2x} \left\{ \frac{e^{2x} \cos 2x - 1}{2} + \frac{e^{2x} \sin 2u}{2} \right\} \bigg|_0^x \] - I.

That is,
\[ 2I = e^{-2x} \left\{ \frac{e^{2x} \cos 2x - 1}{2} + \frac{e^{2x} \sin 2x}{2} \right\} . \]

Therefore,
\[ I = e^{-2x} \left\{ \frac{e^{2x} \cos 2x - 1}{4} + \frac{e^{2x} \sin 2x}{4} \right\} \]
\[ = \frac{\sin 2x + \cos 2x - e^{-2x}}{4} . \]

and
\[ \mathcal{L}^{-1} \left\{ \frac{p}{(p + 2)(p^2 + 4)} \right\} = \frac{\sin 2x + \cos 2x - e^{-2x}}{4} . \]

Hence, the solution to the given differential equation subject to the given initial condition is
\[ y(x) = e^{-2x} + \frac{\sin 2x + \cos 2x - e^{-2x}}{4} . \]

(c) Solve the integral equation for \( y(t) \) where
\[ y(t) = 1 + \int_0^t y(u) \, du. \]
Method

Let \( \mathcal{L} \{ y(t) \} = Y(p) \) then the Laplace transform of the integral equation becomes

\[
Y(p) = \frac{1}{p} + \mathcal{L} \left\{ \int_{0}^{t} y(u) \, du \right\}.
\]

Now

\[
\mathcal{L} \left\{ \int_{0}^{t} y(u) \, du \right\} = \mathcal{L} \left\{ \int_{0}^{t} 1 \cdot y(u) \, du \right\} = \mathcal{L} \{ 1 \} \times \mathcal{L} \{ y(u) \} = \frac{1}{p} Y(p).
\]

Therefore, we have

\[
Y(p) = \frac{1}{p} + \frac{1}{p} Y(p).
\]

Upon solving for \( Y(p) \) we obtain that

\[
Y(p) = \frac{1}{p - 1}.
\]

Finding the inverse Laplace transform of \( Y(p) \) we have that

\[
y(t) = \mathcal{L}^{-1} \{ Y(p) \} = e^{t}.
\]

(d) Solve the integro-differential equation for \( f(t) \) where

\[
f'(t) = \sinh 2t - 4 \int_{0}^{t} f(u) \cosh 2(t - u) \, du
\]

subject to \( f(0) = 1 \).

Method

Let \( \mathcal{L} \{ f(t) \} = F(p) \) then

\[
\mathcal{L} \{ f'(t) \} = pF(p) - f(0) = pF(p) - 1
\]

using the given initial condition and

\[
\mathcal{L} \left\{ \int_{0}^{t} f(u) \cosh 2(t - u) \, du \right\} = F(p) \frac{p}{p^2 - 4}
\]

where

\[
g(t - u) = \cosh 2(t - u) \quad \Rightarrow \quad g(u) = \cosh 2u
\]

\[
\Rightarrow \quad G(p) = \mathcal{L} \{ g(u) \} = \frac{p}{p^2 - 4}.
\]

Also, \( \mathcal{L} \{ \sinh 2t \} = \frac{2}{p^2 - 4} \). Therefore,

\[
pF(p) - 1 = \frac{2}{p^2 - 4} - 4 \left\{ F(p) \frac{p}{p^2 - 4} \right\}.
\]

That is,

\[
\left( p + \frac{4p}{p^2 - 4} \right) F(p) = 1 - \frac{2}{p^2 - 4}
\]
Upon simplifying we have \[
\frac{p^3}{p^2 - 4} F(p) = 1 + \frac{2}{p^2 - 4}.
\]
Upon solving for \(F(p)\) we have \[
F(p) = \frac{1}{p} - \frac{2}{p^3}.
\]
Therefore, using the Laplace transform tables we find that \[
f(t) = 1 - t^2.
\]

**Exercise 5D**

1. Use Laplace transforms to solve the following differential equations.

   (a) \(x'' + 7x' + 12x = 12\)
   where \(x(0) = 0, \ x'(0) = 5\)

   (b) \(x'' + 3x' + 2x = 2e^{-2t}\)
   where \(x(0) = 1, \ x'(0) = -5\)

   (c) \(x'' + 2x' + 2x = 0\)
   where \(x(0) = 1, \ x'(0) = 0\)

   (d) \(x'' + x' - 2x = \cos t + 2\sin t\)
   where \(x(0) = 1, \ x'(0) = 1\)

2. Use the Convolution theorem to find the inverse transforms of the following functions.

   (a) \(\frac{1}{p(p^2 + 9)}\)

   (b) \(\frac{p}{(p^2 + 1)(p^2 + 4)}\)

   (c) \(\frac{1}{p^2(p^2 - 4)}\)

   (d) \(\frac{1}{(p^2 + n^2)^2}\)

   \(\text{Hint:} \ \frac{1}{(p^2 + n^2)^2} = \frac{1}{p^2 + n^2} \times \frac{1}{p^2 + n^2}\).

   (e) \(\frac{1}{(p^2 - 4)^2}\)

   (f) \(\frac{s}{(p^2 - 4)^2}\)

3. Given that

   \(\mathcal{L}\{e^{-bt}x(t)\} = X(p + b), \ \text{(Shift Theorem)}\)

   \(\mathcal{L}\{t^n x(t)\} = (-1)^n \frac{d^n}{dp^n} X(p),\)

   when \(n\) is an integer.

   Find the following Laplace transforms:

   (a) \(t^n\) \quad (b) \(t \cos nt\)

   (c) \(t \sin nt\) \quad (d) \(t^n e^{-bt}\)

   (e) \(t^n e^{-bt}\) \quad (f) \(t^2 \cos 3t\)

   (g) \(t^3 \sin 2t\) \quad (h) \(e^{4t} \cos t\)

   (i) \(e^{-2t} \sin t\)

4. (a) If \(F(p)\) and \(G(p)\) are the Laplace transforms of \(f(x)\) and \(g(x)\), prove that

   \[
   \int_0^\infty F(p)g(p) \, dp = \int_0^\infty f(x)G(x) \, dx,
   \]

   and hence in particular that

   \[
   \int_0^\infty f(x) \frac{dx}{x} = \int_0^\infty F(p) \, dp.
   \]

   (b) Hence evaluate the following integrals.

   (i) \(\int_0^\infty \frac{\sin ax}{x} \, dx, \ a > 0\).

   (ii) \(\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx\).

   continued next page...
5. Solve the following Volterra integral equations.
   (a) \( g(x) = \sin x + \int_0^x \sin(x - u)g(u) \, du \).
   (b) \( g(x) = e^x + 2 \int_0^x \cos(x - t)g(t) \, dt \).

6. Solve the following integro-differential equations.
   (a) \( \int_0^x y(u) \, du - y'(x) = x \).
   (b) \( f'(x) - k^2 \int_0^x f(t) \cos k(x - t) \, dt = 0 \).

7. Use Laplace transforms to show that
   \( y = \frac{1}{k} \int_0^x f(u) \sin k(x - u) \, du \)
   is a solution of
   \[ \frac{d^2 y}{dx^2} + k^2 y = f(x), \]
   when \( y(0) = y'(0) = 0 \).

8. By taking the Laplace transform of the integral
   \( \int_0^x u^{m-1}(x-u)^{n-1} \, du \),
   using the convolution theorem, and then letting \( x = 1 \), show that
   \( B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \).

9. By taking transforms with respect to the parameter \( t \), evaluating the resultant integral and then inverting, show that
   (a) \( \int_{-\infty}^{\infty} \frac{\sin tx^2}{x} \, dx = \int_0^{\infty} \frac{1 - \cos 2tx}{x^2} \, dx = \pi t \)
   and
   (b) \( \int_0^{\infty} e^{-tx^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{t}} \).

5.7 SYSTEMS OF EQUATIONS

Laplace transforms can be used to transform a set of linear system of differential equations with initial conditions into a set of algebraic equations.

Example

Use Laplace transforms to solve the following initial value problem:

\[ \frac{d^2 x}{dt^2} = -10x + 4y \]
\[ \frac{d^2 y}{dt^2} = 4x - 4y \]

subject to \( x = 0, \ y = 0, \ \frac{dx}{dt} = 1 \) and \( \frac{dy}{dt} = -1 \) when \( t = 0 \).

Note There are two independent variables \( x, y \) and \( t \) is the dependent variable. Therefore, we take Laplace transforms with respect to the dependent variable, \( t \).

Let \( \mathcal{L}\{x(t)\} = X(p) \) and \( \mathcal{L}\{y(t)\} = Y(p) \). Hence, the system of differential equations becomes a system of algebraic equations, namely,
\[ p^2 X - px(0) - x'(0) = -10X + 4Y \]
\[ p^2 Y - py(0) - y'(0) = 4X - 4Y \]

where \( x(0) = 0 \), \( x'(0) = 1 \), \( y(0) = 0 \) and \( y'(0) = -1 \).

Upon substituting the initial conditions, the algebraic equations reduce to
\[
\begin{align*}
(p^2 + 10)X - 4Y &= 1 \\
(p^2 + 4)Y - 4X &= -1
\end{align*}
\]

*Note* the function of \( p \) is dropped here purely for convenience.

Solving these equations simultaneously, we find that
\[
\begin{align*}
X(p) &= \frac{p^2}{(p^2 + 2)(p^2 + 12)} \\
Y(p) &= -\frac{p^2 + 6}{(p^2 + 2)(p^2 + 12)}
\end{align*}
\]

We need now find \( x(t) = \mathcal{L}^{-1}(X(p)) \) and \( y(t) = \mathcal{L}^{-1}(Y(p)) \). To do this we have to use partial fractions. Therefore,  
\[
\begin{align*}
X(p) &= -\frac{1}{p^2 + 2} + \frac{6}{p^2 + 12} \\
Y(p) &= -\frac{2}{p^2 + 2} - \frac{3}{p^2 + 12}
\end{align*}
\]

Hence,
\[
\begin{align*}
x(t) &= \mathcal{L}^{-1}(X(p)) = \mathcal{L}^{-1}\left\{-\frac{1}{p^2 + 2} + \frac{6}{p^2 + 12}\right\} \\
&= -\frac{\sqrt{2}}{10} \sin \sqrt{2} t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3} t
\end{align*}
\]

and
\[
\begin{align*}
y(t) &= \mathcal{L}^{-1}(Y(p)) = \mathcal{L}^{-1}\left\{-\frac{2}{p^2 + 2} - \frac{3}{p^2 + 12}\right\} \\
&= -\frac{\sqrt{2}}{5} \sin \sqrt{2} t + \frac{\sqrt{3}}{10} \sin 2\sqrt{3} t
\end{align*}
\]

The general solution to the initial value problem can be now written in the following form:
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{10} \sin \sqrt{2} t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3} t \\ -\frac{\sqrt{2}}{5} \sin \sqrt{2} t + \frac{\sqrt{3}}{10} \sin 2\sqrt{3} t \end{pmatrix}.
\]
**Exercise 5E**

1 Use Laplace transforms to solve the sets of simultaneous differential equations:
   
   (a) \[ \frac{dx}{dt} + 2x + 3y = 0; \quad \frac{dy}{dt} - 3x + 2y = 0; \]
   \[ x = 1, y = 0 \quad \text{when} \quad t = 0 \]
   
   (b) \[ \frac{dy}{dt} - 2x + y = 0; \quad 2 \frac{dx}{dt} + y = 4 \cos t; \]
   \[ y = 0, x = 2 \quad \text{when} \quad t = 0 \]

2 Use Laplace transforms to solve the following initial value problem.

   (a) \[ \frac{dx}{dt} = -x + y \]
   \[ \frac{dy}{dt} = 2x \]
   
   where \( x(0) = 0 \) and \( y(0) = 1 \).

   (b) \[ \frac{dx}{dt} = 2y + e^t \]
   \[ \frac{dy}{dt} = 8x - t \]

   where \( x(0) = 1 \) and \( y(0) = 1 \).

(c) \[ \frac{d^2 x}{dt^2} + x - y = 0 \]
   \[ \frac{d^2 y}{dt^2} + y - x = 0 \]

   where \( x(0) = 0, x'(0) = -2 \)
   and \( y(0) = 1, y'(0) = 1 \).

(d) \[ \frac{dx}{dt} = 4x - 2y + 2h(t-1) \]
   \[ \frac{dy}{dt} = 3x - y + h(t-1) \]

   where \( x(0) = 0 \) and \( y(0) = \frac{1}{2} \).
<table>
<thead>
<tr>
<th>( y(t) )</th>
<th>( Y(p) = \int_0^{\infty} y(t) \ e^{-pt} \ dt )</th>
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<tr>
<td>1</td>
<td>( \frac{1}{p} )</td>
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<tr>
<td>( t^r, \ r &gt; -1 )</td>
<td>( \frac{\Gamma(r + 1)}{p^{r+1}} )</td>
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<td>( e^{-bt} )</td>
<td>( \frac{1}{p + b} )</td>
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<tr>
<td>( \sin nt )</td>
<td>( \frac{n}{p^2 + n^2} )</td>
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<td>( \cos nt )</td>
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<td>( \sinh nt )</td>
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<td>( \cosh nt )</td>
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<td>( e^{-bt} f(t) )</td>
<td>( F(p + b) )</td>
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<td>( h(t - a) )</td>
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<td>( f(t - a)h(t - a) )</td>
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<td>( f'(t) )</td>
<td>( pF(p) - f(0) )</td>
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<tr>
<td>( f''(t) )</td>
<td>( p^2 F(p) - pf(0) - f'(0) )</td>
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<tr>
<td>( \int_0^t f(t - u) \ g(u)du )</td>
<td>( F(p) \ G(p) )</td>
</tr>
<tr>
<td>( t^n f(t) )</td>
<td>( (-1)^n \frac{d^n}{dp^n} F(p) )</td>
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Chapter 6: Fourier Series

6.1 INTRODUCTION

Fourier Analysis breaks down functions into their frequency components. In particular, a function can be written as a series involving trigonometric functions. For example, let \( f(t) \) be a function defined on the interval \( a < t < a + 2l \) then \( f(t) \) can be written in terms of a series expansion. That is,

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi t}{l} \right) + b_n \sin \left( \frac{n\pi t}{l} \right) \right)
\]

where \( a_0, a_n \) and \( b_n \) are called the co-efficients which are constant and are to be determined. The frequency components are \( \frac{n\pi}{l} \) for \( n = 1, 2, \ldots \).

In many instances, the magnitude of the co-efficients (i.e. \( \sqrt{a_n^2 + b_n^2} \)) is graphed against the frequencies. This graph is called the spectral graph of the function and the named given to the analysis of data is called spectral analysis. Engineers use spectral analysis to determine the dominant frequency for a particular problem.

For example, consider slender column of uniform cross-section as described in the figure below.

Here \( y(x) \) is the deflection curve for the column. Therefore, \( y(x) \) satisfies

\[
\frac{d^2 y}{dx^2} + \frac{P}{EI} y = 0
\]

where \( P \) is the load on the column, \( E \) is Young’s modulus and \( I \) is the moment of inertia for the column.

If the column is constrained at both ends then we can apply the boundary conditions

\[
y(0) = y(L) = 0.
\]

It can be shown that the deflection curve of the column is

\[
y(x) = \sum_{1}^{\infty} y_n(x)
\]

\[
= \sum_{1}^{\infty} c_n \sin \sqrt{P_n} x
\]
where
\[ P_n = \frac{n^2 \pi^2}{L^2} EI, \quad \text{where } n = 1, 2, 3, \ldots. \]

This particular problem is called the eigenvalue problem and will be later discussed in the next chapter. It is easily seen that \( y(x) \) is now in terms of trigonometric functions. The particular series is called Fourier Series. Also, the ‘frequency’ component is \( P_n \), which is called the critical loads. When \( n = 1 \),
\[ P_1 = \frac{\pi^2}{L^2} EI. \]

This is called the Euler load or dominant load. Also,
\[ y_1(x) = c_1 \sin \frac{\pi x}{L} \]
is called the first buckling load.

### 6.2 FUNCTIONS

Before the introduction to Fourier Series it is worthwhile refreshing some ideas and concepts about functions. In particular, the properties of odd and even functions. We shall use these properties quite frequently when discussing Fourier Series later on.

#### 6.2.1 Properties of Even Functions

Recall that if \( f(x) \) is an even function then
\[ f(-x) = +f(x). \]

(a) The graph of an even function is symmetric about the y-axis (or vertical).

Examples

Figure (a) could be expressed as the function rule \( f(x) = x^4 - x^2 \), Figure (b) with rule \( f(x) = (2 - x)^2 \) for \( 0 < x < 2 \) and Figure (c) with rule \( f(x) = 2 \) for \( x \in [-a, a] \). It can be seen from these graphs that even functions have the property that
\[ \int_{-a}^{a} f(x) \, dx = 2 \int_0^{a} f(x) \, dx. \]

(b) The Maclaurin series of an even function contains even powers of \( x \) only.

Examples of even functions: \( 1, x^2, x^4, \cos x, \cosh x, \ldots. \)
6.2.2 Properties of Odd Functions

Recall that if \( f(x) \) is an odd function then

\[ f(-x) = -f(x). \]

(a) The graph of an odd function is skew-symmetric through the origin.

Examples

The function rule for Figure (a) could be written as \( f(x) = x^3 \), whereas Figure (b) could have the rule \( f(x) = x \). Figure (c) could be written in function form as \( f(x) = 1 \) for \( x > 0 \) and \( f(x) = -1 \) for \( x < 0 \).

Odd functions have the property that

\[ \int_{-a}^{a} f(x) \, dx = 0. \]

(b) Odd functions have the property that their Maclaurin series contains odd powers of \( x \) only.

Examples of odd functions: \( x, x^3, \sin x, \sinh x, \text{erf}(x) \), . . .

(c) If \( f(x) \) is continuous at \( x = 0 \) then \( f(0) = 0 \).

6.2.3 Products of Odd and Even Functions

The property of the product of even and odd functions can be summarised as follows:

\[ \text{(even function)} \times \text{(even function)} = \text{(even function)} \]
\[ \text{(odd function)} \times \text{(odd function)} = \text{(even function)} \]
\[ \text{(even function)} \times \text{(odd function)} = \text{(odd function)}. \]

Example

The function \( x \) is an odd function and \( \cos x \) is an even function, and hence their product is an even function. That is,

\[ x \cos x = \text{(odd function)} \times \text{(even function)} = \text{(odd function)}. \]

Thus,

\[ \int_{-2}^{2} x \cos x \, dx = 0. \]
Similarly, 

\[ x \sin x = (\text{odd function}) \times (\text{odd function}) = (\text{even function}) \]

Thus,

\[ \int_{-2}^{2} x \sin x \, dx = 2 \int_{0}^{2} x \sin x \, dx. \]

### 6.2.4 Trigonometry Formulae

\[
\begin{align*}
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\sin(A - B) &= \sin A \cos B - \cos A \sin B \\
\cos(A - B) &= \cos A \cos B + \sin A \sin B \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\
2 \cos A \sin B &= \sin(A + B) - \sin(A - B) \\
2 \cos A \cos B &= \cos(A - B) + \cos(A + B) \\
2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\
\sin 2A &= 2 \sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2 \cos^2 A - 1 \\
&= 1 - 2 \sin^2 A
\end{align*}
\]

*Note:*

When \( n \) is an integer

\[ \sin n\pi = 0 \quad \text{and} \quad \cos n\pi = (-1)^n. \]

### 6.2.5 Orthogonal Functions

Two functions \( f_m \) and \( f_n \) are said to be orthogonal on an interval \([a, b]\) if

\[ \int_{a}^{b} f_m(x) f_n(x) \, dx = 0. \]

Examples of such orthogonal functions are the trigonometric functions Sine and Cosine and the Bessel function. In particular, the set \( \{1, \cos x, \cos 2x, \ldots\} \) form a set of orthogonal functions.

Thus if \( m \) and \( n \) are integers, then the following integration properties hold for both the sine and cosine functions:

\[
\begin{align*}
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= 0 \\
\int_{-\pi}^{\pi} \cos mx \sin nx \, dx &= 0 \\
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= 0 \\
\int_{-\pi}^{\pi} \cos^2 mx \, dx &= \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi, \quad m \neq 0.
\end{align*}
\]
6.2.6 Periodic Functions

A function \( f(t) \) is called periodic if it is defined for all real \( t \) and if there is a positive number \( \tau \) that

\[
f(t + \tau) = f(t).
\]

This number \( \tau \) is called the period of \( f(t) \). The graph of such a function is obtained by the periodic repetition of its graph in any interval of length \( \tau \), as shown in the figures below.

The trigonometric functions are the most common periodic functions of period \( 2\pi \).

*Note:* The function \( f(t) = c = \text{constant} \) is a periodic function.

6.2.7 Sinusoids

Sometimes a function occurs as a linear combination of sine functions or sinusoids. For instance, consider the function

\[
f(t) = a \sin \omega t + b \sin 2\omega t + c \sin 3\omega t + d \sin 4\omega t.
\]

The fundamental angular frequency is \( \omega \) as this is the lowest frequency (or lowest valued co-efficient of \( t \)) for each of the sin terms. Thus the frequency of oscillation is defined as

\[
\frac{\omega}{2\pi}
\]

which is usually measured in cycles per second or Hertz. Hence, the largest period of the sin terms is

\[
\text{Period} = \tau = \frac{2\pi}{\omega}.
\]

The other frequencies are integer multiples of this lowest frequency. Thus the fundamental or first harmonic is the term that involves the fundamental frequency. That is, \( a \sin \omega t \) is the fundamental or first harmonic. The next term, \( b \sin 2\omega t \) is called the second harmonic, and so on.

The amplitude of the first harmonic is \( a \) and the amplitude of the second harmonic is \( b \).
Note: Recall the trigonometric identity that

\[ R \sin(\omega t + \phi) = R \sin \omega t \cos \phi + R \cos \omega t \sin \phi \]

\[ = A \sin \omega t + B \cos \omega t \]

where

\[ A = R \cos \phi \quad B = R \sin \phi. \]

Therefore,

\[ R = \sqrt{A^2 + B^2} \]

\[ \phi = \tan^{-1} \frac{B}{A} \]

Thus a function that has a linear combination of sine’s and cosine’s can be expressed as a sum of sinusoids with different amplitudes and phase angles. That is,

\[ f(t) = \sum_{n=1}^{\infty} a_n \sin n \omega t + b_n \cos n \omega t \]

\[ = \sum_{n=1}^{\infty} R_n \sin (n \omega t + \phi_n) \]

Example

(a) Describe the frequency and amplitude characteristics of the different components of the function

\[ f(t) = \sin 3\pi t - 0.7 \sin 9\pi t + 0.3 \sin 15\pi t. \]

Method

The fundamental angular frequency is \( 3\pi \) or frequency of oscillation is \( \frac{3}{2} \) hertz.

The amplitude of the first harmonic is 1.

The fundamental or first harmonic is \( \sin 3\pi t \).

The second and forth harmonics are missing, that is, the terms \( \sin 6\pi t \) and \( \sin 12\pi t \).

The third harmonic amplitude is 0.7

(b) If \( f(t) = \sin t + 2 \cos t \), express \( f(t) \) as a single sinusoid and hence determine its amplitude and phase.

Method

Recall the trigonometric identity then

\[ R = \sqrt{1^2 + 2^2} = \sqrt{5} \]

\[ \phi = \tan^{-1} \frac{2}{1} = \tan^{-1} 2 \]

\[ \approx 1.1 \text{ rad.} \]

Hence,

\[ f(t) = \sqrt{5} \sin(t + 1.1). \]

Thus the amplitude is \( \sqrt{5} \) and the phase angle is \( \approx 1.1 \) radians.
Note: Our function \( f(t) \) could have been in terms of the \( \cos \) function rather than the \( \sin \) function. We could use the same techniques as above except that our phase angle would differ by \( \frac{\pi}{2} \).

### 6.2.8 Laplace Transform of a Periodic Function

As an engineer or an applied mathematician, it is important to know about periodic functions and their behaviour. This is due to the fact that these functions arise in many practical problems. Periodic functions can be more complicated than the normal periodic functions of sines and cosines. One important aspect is knowing how to take Laplace transforms of periodic functions.

If \( f(t) \) is a piecewise continuous function of interval length \( \tau \), then its Laplace transform exists. That is,

\[
\mathcal{L} \{ f(t) \} = \int_0^\infty e^{-st} f(t) \, dt.
\]

We can rewrite the integral on the right-hand side into a series of integrals over successive periods, namely,

\[
\mathcal{L} \{ f(t) \} = \int_0^\tau e^{-st} f(t) \, dt + \int_\tau^{2\tau} e^{-st} f(t) \, dt + \int_{2\tau}^{3\tau} e^{-st} f(t) \, dt + \ldots.
\]

Substituting \( z = t - \tau \) in the second integral, \( z = t - 2\tau \) in the third integral and, consequently, \( z = t - n\tau \) for the \( n \)th integral etc, then integrals become

\[
\mathcal{L} \{ f(t) \} = \int_0^\tau e^{-sz} f(z) \, dz + \int_0^\tau e^{-s(z+\tau)} f(z) \, dz + \int_0^\tau e^{-s(z+2\tau)} f(z) \, dz + \ldots
\]

\[
= (1 + e^{-s\tau} + e^{-2s\tau} + \ldots) \int_0^\tau e^{-sz} f(z) \, dz.
\]

Hence, by the binomial expansion we find that

\[
\mathcal{L} \{ f(t) \} = \frac{1}{1 - e^{-s\tau}} \int_0^\tau e^{-sz} f(z) \, dz.
\]

**Example**

Find the Laplace transform \( f \) where \( f(t) \) is the square wave as shown.

![Square Wave Diagram](image-url)
Here $\tau = b$ then

\[
\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sb}} \int_0^b e^{-sz} f(z) \, dz
\]

\[
= \frac{1}{1 - e^{-sb}} \left( \int_0^a e^{-sz} f(z) \, dz + \int_a^b e^{-sz} f(z) \, dz \right)
\]

\[
= \frac{1}{1 - e^{-sb}} \int_a^b e^{-sz} \, dz = \frac{e^{-sa} - e^{-sb}}{s(1 - e^{-sb})}
\]

**Exercise 6A**

1. Determine if the following functions are even or odd. Sketch their graphs.

   (a) $f(x) = -x$

   (b) $f(x) = \frac{x}{2} + 1$

   (c) $f(x) = x^3$

   (d) $f(x) = \frac{x}{2} + \sin x$

   (e) $f(x) = e^{-x^2}$

   (f) $f(x) = |x|

2. State whether the following functions are even, odd or neither.

   (a) $f(x) = x^2 \sin x$

   (b) $f(x) = x^2 \sin 2x$

   (c) $f(x) = \sin 3x \sin 2x$

   (d) $f(x) = x^2 \sin 2x$

   (e) $f(x) = e^{-x^2} \sin x$

   (e) $f(x) = x|x|

3. Prove the following orthogonality results for trigonometric functions.

   (a) $\int_{-\pi}^{\pi} \sin mx \, dx = 0$

   (b) $\int_{-\pi}^{\pi} \cos mx \, dx = \begin{cases} 0, & m \neq 0 \\ 2\pi, & m = 0 \end{cases}$

   (c) $\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$

   (d) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \text{ and } m \neq 0 \\ 2\pi, & m = n = 0 \end{cases}$

   (e) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \text{ and } m \neq 0 \\ 0, & m = n = 0 \end{cases}$

   (f) $\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi, \ m \neq 0$

4. Sketch the graph of the following periodic functions.

   (a) $f(x) = x, -1 \leq x \leq 1$ and

   $f(x + 2) = f(x)$.

   *continued next page...*
5 Express the following functions as a single sinusoid and hence find their amplitudes and phases.

(a) \( f(x) = 3 \sin x - 2 \cos x \)

(b) \( f(x) = 2 \sin 3x \)

(c) \( f(x) = 3 \sin 2x + 2 \cos 2x \)

6 Find the Laplace transform of the following periodic functions that are shown below.

6.3 DEFINITION

An important skill for engineers is to be able to analyse certain wave forms. In particular the frequency components of a wave. A mathematical tool that enables an engineer to break a wave to its frequency components is called Fourier Analysis. This section will look at the essential requirements to break down a wave into its frequency components by using Fourier Series.

6.3.1 Fourier Series

Generally, a Fourier series for a function \( f(x) \) defined on an interval \( a < x < a + 2l \) is of the form

\[
f(x) \sim S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x,
\]

where

\[
a_n = \frac{1}{l} \int_{a}^{a+2l} f(x) \cos \frac{n\pi}{l} x \, dx \quad n = 0, 1, 2, \ldots,
\]

and

\[
b_n = \frac{1}{l} \int_{a}^{a+2l} f(x) \sin \frac{n\pi}{l} x \, dx \quad n = 1, 2, 3, \ldots.
\]

A property of this series is that it will converge to the mean value of the function at any discontinuity either in the interval or at an endpoint.
If the defining function \( f(x) \) is itself only defined on a given interval \( a < x < a + 2l \) (and not extended to be periodic by some part of its definition), then the Fourier series which converges to the function \( f(x) \) in the defining interval may be extended to represent a function \( F(x) \) such that

\[
F(x) = \begin{cases} 
  f(x), & a < x < a + 2l \\
  f(x + 2l), & \text{elsewhere}
\end{cases}
\]

An advantage of studying Fourier series is that the sum of some infinite real series can be obtained. This can be achieved by substituting values of \( x \) into the series and the function. An example of this will be seen later.

Specifically, a **Fourier series** for a function \( f(x) \) defined on an interval \( -\pi < x < \pi \) is of the form

\[
f(x) \sim S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx ,
\]

where

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad n = 0, 1, 2, \ldots,
\]

and

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad n = 1, 2, 3, \ldots.
\]

**Note:**

Here \( a = -\pi \) and \( a + 2l = \pi \), hence \( l = \pi \) and the function is assumed to have period of \( 2\pi \).

**Example**

Find the Fourier series, \( S(x) \), of the function \( f(x) = x \) on the interval \( -\pi < x < \pi \).

**Method**

Here \( a = -\pi \) and \( a + 2l = \pi \). That is, \( l = \pi \). Also, it is assume that the function \( f(x) \) is periodic with period equal to the length of the interval ( \( 2\pi \)).

The Fourier series for \( f(x) \) is

\[
S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos(nx) + b_n \sin(nx) \}
\]

where the co-efficients are

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx ,
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \quad n = 1, 2, 3, \ldots,
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \quad n = 1, 2, 3, \ldots.
\]

At this stage it is beneficial to draw a graph of the function to aid in determining the Fourier co-efficients and therefore the Fourier series. This is due to the fact that some of the co-efficients may be automatically zero and thus the Fourier series would be simplified.
The graph of \( f(x) \) is

From the graph of \( f(x) \), it can be seen that \( f(x) \) is an odd function. Since, the sine function is an odd function, it can be easily shown that

\[
a_0 = a_n = 0
\]

automatically and therefore we need only find the \( b_n \) co-efficient.

Here,

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx \quad \text{(integrate by parts)}
\]

\[
= \frac{-2 \cos(n\pi)}{n}
\]

where \( \cos(n\pi) = (-1)^n \)

\[
= \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}
\]

Thus, the Fourier series for \( f(x) \) is given by

\[
S(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).
\]

**Note:**

This series is an infinite series. We can look at the partial sum of this series to determine how well it approximates the given function. The first few terms of this series are

\[
2 \sin(x) - \sin(2x) + \frac{2}{3} \sin(3x) - \frac{1}{2} \sin(4x) + \ldots
\]

The following figure compares the function \( f(x) \) to the partial sums \( S_1 \), \( S_2 \) and \( S_{10} \), respectively.
It can be seen that the higher number of terms in the partial sum, the better the approximation to the given function.

6.3.2 Periodicity and Discontinuities

Recall \( f(x) \). That is,

\[
f(x) = x \quad -\pi < x < \pi.
\]

Then, let \( F(x) \) be the periodic function of \( f(x) \) then the graph of \( F(x) \) over the interval \((-3\pi, 3\pi)\) is

![Graph of F(x)](image)

It can be seen from the above figure that \( F(x) \) has discontinuities at \(-3\pi, -\pi, \pi, 3\pi, \ldots\).

For instance, when \( x = \pi \) (say), \( F(x) \) is not defined. Also, from the graph of periodic function \( F(x) \) we can see that,

\[
\lim_{x \to \pi^-} F(x) = \pi \quad \text{and} \quad \lim_{x \to \pi^+} F(x) = -\pi
\]

If we consider the Fourier series we find that

\[
\lim_{x \to \pi} S(x) = \lim_{x \to \pi} 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = 0.
\]

**Note:**

\( S(0) \) is the average of the 2 limit values obtained above. That is,

\[
\frac{1}{2} \left\{ \lim_{x \to \pi^-} F(x) + \lim_{x \to \pi^+} F(x) \right\} = 0 = S(\pi).
\]

Generally, at a point of discontinuity of \( F(x) \), the Fourier series averages the limiting values of the function at the discontinuous point. That is, if \( x = a \) be a discontinuous point of \( F(x) \) then

\[
\frac{1}{2} \left\{ \lim_{x \to a^-} F(x) + \lim_{x \to a^+} F(x) \right\} = S(a).
\]
6.3.3 Finding the Sum of a Series

We can use the Fourier series approximation to a given function to determine the sum of an infinite series. For instance, recall that

\[ f(x) = x, \quad -\pi < x < \pi \]

and

\[ f(x) \sim S(x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx). \]

If we let \( x = \frac{\pi}{2} \) (say), then

\[ f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}. \]

Upon substituting \( x = \frac{\pi}{2} \) into the Fourier series we obtain

\[ S\left(\frac{\pi}{2}\right) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right). \]

The infinite series can be simplified further due to the fact that when \( n \) is an even integer \( \sin\left(\frac{n\pi}{2}\right) = 0 \).

That is,

\[ \sin\left(\frac{n\pi}{2}\right) = 0 \quad \text{for} \quad n = 2, 4, 6, \ldots. \]

Also, when \( n \) is an odd integer we can simplify the term \( \sin\left(\frac{n\pi}{2}\right) \) as follows. Firstly, we rewrite the odd values of \( n \) in the form of \( 2n - 1 \) where \( n = 1, 2, 3, \ldots \), then \( \sin\left(\frac{\pi n}{2}\right) \) can be written as

\[
\sin\left(\frac{(2n-1)\pi}{2}\right) = \sin\left(n\pi - \frac{\pi}{2}\right) = \sin n\pi \cos \frac{\pi}{2} - \cos n\pi \sin \frac{\pi}{2} = (-1)^{n+1}.
\]

As a result,

\[ S\left(\frac{\pi}{2}\right) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n-1} (-1)^{n-1} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \quad \text{upon simplification} \]

Since \( f\left(\frac{\pi}{2}\right) \sim S\left(\frac{\pi}{2}\right) \) then

\[ \frac{\pi}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \]

or

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}. \]

Hence, with the aid of Fourier series, we have found the sum of an infinite series.
Example

Find the Fourier series \( S(x) \) for the following function

\[
f(x) = \begin{cases} 
0, & -3 \leq x \leq 0 \\
x, & 0 \leq x \leq 3 
\end{cases}
\]

Here \( a = -3 \) and \( a + 2l = 3 \), therefore, \( l = 3 \). The function \( F(x) \) is assume to be the periodic extension of \( f(x) \) with period 6 (ie the length of the interval). The graph of \( F(x) \) over the interval \([-9,9]\) is

From this graph it can be seen that the function \( f(x) \) is neither an even or odd function. Therefore, all Fourier co-efficient need to be evaluated.

The Fourier co-efficients of \( f(x) \) on the interval \([-3,3]\) are

\[
a_0 = \frac{1}{3} \int_{-3}^{3} f(x) \, dx = \frac{1}{3} \int_{0}^{3} x \, dx = \frac{3}{2}.
\]

and

\[
a_n = \frac{1}{3} \int_{-3}^{3} f(x) \cos \frac{n\pi x}{3} \, dx = \frac{1}{3} \int_{0}^{3} x \cos \frac{n\pi x}{3} \, dx = \frac{3}{n^2\pi^2} \left[ \cos(n\pi) - 1 \right] = \frac{3}{n^2\pi^2} \left( (-1)^n - 1 \right).
\]

Now \( a_n \) can be further simplified due to the fact that when \( n \) is an even integer \( a_n = 0 \). However, when \( n \) is an odd integer \( a_n = \frac{-6}{n^2\pi^2} \). That is, the odd co-efficients of \( a_n \) can be expressed as

\[
a_{2n-1} = \frac{-6}{(2n-1)^2\pi^2} \quad \text{for} \quad n = 1, 2, 3, \ldots.
\]
Now
\[ b_n = \frac{1}{3} \int_{-3}^{3} f(x) \sin \frac{n\pi x}{3} \, dx \]
\[ = \frac{1}{3} \int_{0}^{3} x \sin \frac{n\pi x}{3} \, dx \]
\[ = -\frac{3}{n\pi} \cos(n\pi) \quad \text{cos}(n\pi) = (-1)^n \]
\[ = \frac{3}{n\pi}(-1)^{n+1}. \]

Therefore the Fourier series of \( f(x) \) on the interval \([-3, 3]\) is
\[ S(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \left( \frac{-6}{(2n-1)^2\pi^2} \cos \left( \frac{(2n-1)\pi x}{3} \right) + \frac{3}{n\pi}(-1)^{n+1} \sin \left( \frac{n\pi x}{3} \right) \right). \]

**Exercise 6B**

1. For each of the functions \( f(x) \) below, let \( F(x) \) be the periodic extension of \( f(x) \) then

   (i) Sketch the graph of the function \( f(x) \) on \((-\pi, \pi)\).

   (ii) Find the Fourier series, \( S(x) \), for the function on \((-\pi, \pi)\).

   (iii) Sketch the graph of the function \( F(x) \) (which is represented by the series \( S(x) \)) on the interval \((-5\pi, 5\pi)\).

   (iv) Investigate the identities obtained using

   \[ \frac{1}{2} \left( \lim_{x \to \pi^-} F(x) + \lim_{x \to \pi^+} F(x) \right) = S(\pi). \]

   That is, show that at the point of discontinuity (here \( x = \pi \)), the Fourier series takes the average of the 2 limit values of \( F(x) \).

   The functions are:

   (a) \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \)

   (b) \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases} \)

   (c) \( f(x) = |x|, \quad -\pi < x < \pi \)

   (d) \( f(x) = x^2, \quad -\pi < x < \pi \)

   (e) \( f(x) = |\sin x|, \quad -\pi < x < \pi \).

   (f) \( f(x) = h(x), \quad -1 < x < 1 \).

2. Find the Fourier series for each of the following periodic functions \( f(x) \).

   In each case, make sketches of the functions represented by the series \( S(x) \) on the interval \(-6 < x < 6\).

   (a) \( f(x) = \begin{cases} -1, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases} \)

   where \( f(x+2) = f(x) \)

   (b) \( f(x) = \cos x, \quad 0 < x < 3 \)

   where \( f(x+3) = f(x) \).

3. For each of the given functions in (1), let \( x = \frac{\pi}{2} \) then find the sum of an infinite series.
6.4 FOURIER SINE AND COSINE SERIES

Sometimes it is possible to represent a function as a Fourier Sine or Cosine series. To do this we use the properties of even and odd functions as defined in section 6.1.1. To determine a series we usually extend the interval of definition (backwards) to create a new function that is either even or odd depending on the type of series required. If we require a Fourier sine series then the new function that is created is chosen to be an odd function. Similarly, if we require a Fourier cosine series then the new function created is chosen to be an even function.

For example, let $f(x)$ be defined on the interval $[0, l]$.

(a) If we require a Fourier sine series then we create a new function, $g(x)$ (say), which is an odd function over the interval $[-l, l]$. That is, we let

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq l \\ -f(-x) & -l \leq x \leq 0 \end{cases}.$$ 

We call $g$ an odd extension of $f$ to $[-l, l]$. The Fourier series for $g(x)$ over the interval $[-l, l]$ is a Fourier sine series due to the fact that $g(x)$ is an odd function. Hence, the Fourier coefficients are

$$a_0 = a_n = 0$$

and

$$b_n = \frac{1}{l} \int_{-l}^{l} g(x) \sin \left( \frac{n\pi x}{l} \right) \, dx = \frac{2}{l} \int_{0}^{l} f(x) \sin \left( \frac{n\pi x}{l} \right) \, dx, \quad n = 1, 2, 3, \ldots.$$ 

as $g(x) = f(x)$ over the interval $[0, l]$.

The construction of $g(x)$ is simply an aid to determine the required Fourier series for $f(x)$ on $[0, l]$.

(b) If we require a Fourier cosine series then we create a new function, $g(x)$ (say), which is an even function over the interval $[-l, l]$. That is, we let

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq l \\ f(-x) & -l \leq x \leq 0 \end{cases}.$$ 

We call $g$ an even extension of $f$ to $[-l, l]$. The Fourier series for $g(x)$ over the interval $[-l, l]$ is a Fourier cosine series due to the fact that $g(x)$ is an even function. Hence, the Fourier coefficients are $b_n = 0$ and

$$a_0 = \frac{1}{l} \int_{-l}^{l} g(x) \, dx = \frac{2}{l} \int_{0}^{l} f(x) \, dx.$$ 

and

$$a_n = \frac{1}{l} \int_{-l}^{l} g(x) \cos \left( \frac{n\pi x}{l} \right) \, dx = \frac{2}{l} \int_{0}^{l} f(x) \cos \left( \frac{n\pi x}{l} \right) \, dx, \quad n = 1, 2, 3, \ldots.$$ 

as $g(x) = f(x)$ over the interval $[0, l]$.

Once again, the construction of $g(x)$ is just an aid to determine the required Fourier series for $f(x)$ on $[0, l]$.
6.4.1 Fourier Cosine Series

If \( f(x) \) is a function defined on the interval \( 0 < x < l \), then it has a Fourier Cosine series

\[
f(x) \sim S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}\]

where

\[
a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad n = 0, 1, 2, \ldots .
\]

**Example**

Find the Fourier cosine series for the function \( f(x) = 2x \) where \( 0 \leq x \leq l \).

**Method**

We make an even extension of \( f \) to form the new function \( g \) so that

\[
g(x) = \begin{cases} 
  f(x) = 2x & 0 \leq x \leq l \\
  f(-x) = -2x & -l \leq x \leq 0
\end{cases}
\]

Graphically, we have

Since \( g(x) \) is an even function then the Fourier co-efficients are

\[
b_n = 0,
\]

\[
a_0 = \frac{1}{l} \int_{-l}^{l} g(x) \, dx = \frac{2}{l} \int_{0}^{l} f(x) \, dx
\]

\[
= \frac{2}{l} \int_{0}^{l} 2x \, dx = 2l
\]

and

\[
a_n = \frac{1}{l} \int_{-l}^{l} g(x) \cos \left( \frac{n\pi x}{l} \right) \, dx = \frac{2}{l} \int_{0}^{l} f(x) \cos \left( \frac{n\pi x}{l} \right) \, dx
\]

\[
is an even function on \([-l, l]\)
That is, \( a_n = \frac{2}{l} \int_0^l 2x \cos \left( \frac{n\pi x}{l} \right) \, dx \) integrating by parts
\[
= \frac{4l}{n^2\pi^2} [\cos n\pi - 1] \quad \cos n\pi = (-1)^n
\]
\[
= \frac{4l}{n^2\pi^2} [(-1)^n - 1]
\]
The co-efficient \( a_n \) can be further simplified by noting that when \( n \) is an even integer \( a_n = 0 \). Also, when \( n \) is an odd integer we find that \( a_n = -\frac{8l}{n^2\pi^2} \). We can rewrite the odd co-efficients in the form
\[
a_{2n-1} = -\frac{8l}{(2n-1)^2\pi^2} \quad n = 1, 2, 3, \ldots.
\]
Hence, the Fourier cosine series for \( f(x) \) on the interval \( [0, l] \) is
\[
S(x) = l - \frac{8l}{\pi^2} \sum_{n=1}^\infty \frac{1}{(2n-1)^2\pi^2} \cos \left( \frac{(2n-1)\pi x}{l} \right).
\]

Note: The \( a_0 \) term is divided by 2.

### 6.4.2 Fourier Sine Series

If \( f(x) \) is a function defined on the interval \( 0 < x < l \), then it has a **Fourier Sine series**
\[
f(x) \sim S(x) = \sum_{n=1}^\infty b_n \sin \left( \frac{n\pi x}{l} \right)
\]
where
\[
b_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) \, dx, \quad n = 0, 1, 2, \ldots.
\]

**Example**

Find the Fourier sine series for the function
\[
f(x) = 2x \quad \text{where} \quad 0 \leq x \leq l.
\]

**Method**

We make an odd extension of \( f \) to form the new function \( g \) so that
\[
g(x) = \begin{cases} 
  f(x) = 2x & 0 \leq x \leq l \\
  -f(-x) = 2x & -l \leq x \leq 0.
\end{cases}
\]
Graphically, we have

Since \( g(x) \) is an odd function then the Fourier co-efficients are

\[
a_0 = 0 \quad \text{and} \quad a_n = 0
\]

and

\[
b_n = \frac{1}{l} \int_{-l}^{l} g(x) \sin\left(\frac{n\pi x}{l}\right) \, dx = \frac{2}{l} \int_{0}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) \, dx
\]

\[
= \frac{2}{l} \int_{0}^{l} 2x \sin\left(\frac{n\pi x}{l}\right) \, dx \quad \text{integrating by parts}
\]

\[
= (-1)^{n+1} \frac{4l}{n\pi}.
\]

Therefore, the Fourier sine series, \( S(x) \), for \( f(x) \) on \([0, l]\) is

\[
f(x) \approx S(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)
\]

\[
= \frac{4l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right).
\]

**Note**

The Fourier Cosine and Sine series represent, respectively, an even and an odd function on the interval \(-l < x < l\), which correspond to the defined function \( f(x) \) on the ‘half interval’ \( 0 < x < l \). These series are often called *half interval expansions*.

**Exercise 6C**

1. (a) Expand each of the following functions as indicated over the interval given.

   Also, for each case, make sketches to show the function \( F(x) \) (the periodic extension of \( f(x) \)) which is represented by your series \( S(x) \) on \(-3\pi < x < 3\pi\).

   (i) \( f(x) = 1 \) as a sine series on \( 0 < x < \pi \).

   (ii) \( f(x) = x \) as a cosine series on \( 0 < x < \pi \).

   (iii) \( f(x) = \cos x \) as a sine series on \( 0 < x < \pi \).

   (continued next page...)
(b) Find the identities which occur when you make the following substitutions in your answers to part (a) of this question.

(i) $x = \frac{\pi}{2}$ in (a)(i)

(ii) $x = 0$ in (a)(ii).

(iii) $x = \pi$ in (a)(iii)

(iv) $x = \frac{\pi}{4}$ in (a)(iii).

2 The function $f$ is defined on the interval $(0, 1)$ by $f(x) = 1, \ 0 < x < 1$.

(a) Show that its Fourier sine series is

$$\frac{4}{\pi} \left\{ \sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x + \ldots \right\}$$

(b) On the interval $(-3, 3)$, sketch the function represented by the series in (a).

(c) What series identity can be found by substituting $x = \frac{1}{2}$ in the series in (a)?

3 The function $f$ is defined on the interval $(0, 1)$, by $f(x) = x, \ 0 < x < 1$.

(a) Find its Fourier cosine series.

(b) On the interval $(-3, 3)$, sketch the function represented by the series in (a).

(c) Use the series in (a), or otherwise, to prove that

$$\frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{(2n - 1)^2} + \ldots = \frac{\pi^2}{8}.$$
Chapter 7: Boundary Value Problems

7.1 THE EIGENVALUE PROBLEM

If a general solution is sought for a given differential equation over the interval $0 \leq x \leq l$ (say) with conditions given at both ends of the interval then these conditions are termed boundary conditions. Thus the name given to describe the problem of finding a solution to a given differential equation subject to these boundary conditions is called the Boundary Value Problem. Similarly, the Initial Value Problem is the name given to describe the problem of finding the solution of a given differential equation subject to initial conditions.

Orthogonal functions often arise in the solution of ordinary differential equations. In particular, a set of orthogonal functions can be generated by solving a boundary value problem involving a second-order differential equation containing a parameter $\lambda$. Upon solving the differential equation and using the associated boundary conditions, the parameter $\lambda$ takes on an infinite number of solutions which are called eigenvalues. Each eigenvalue has a corresponding solution to the differential equation which is called an eigenfunction. The sum of these eigenfunctions is the general solution to the given boundary value problem. This is type of boundary value problem is called the Eigenvalue Problem.

Before continuing to explain the Eigenvalue Problem a fundamental property of linear operators has to be stated. This is the Principle of Superposition.

7.1.1 Principle of Superposition

If $u_1$ and $u_2$ satisfies a linear homogeneous equation, then an arbitrary linear combination of them, $c_1u_1 + c_2u_2$, also satisfies the same linear homogeneous equation.

7.1.2 Method of Solution

Example

Solve the following eigenvalue problem:

$$y'' + \lambda y = 0$$

subject to

$$y(0) = 0 \quad \text{the boundary condition at } x = 0$$

$$y(l) = 0 \quad \text{the boundary condition at } x = l$$

The type of solution to the given differential equation will depend on the sign of the constant $\lambda$. Hence, there are three cases to consider. These are:

- CASE I: $\lambda = 0$
- CASE II: $\lambda < 0$
- CASE III: $\lambda > 0$
For each one of the above cases we are seeking non-trivial or non-zero solutions.

**CASE I** When \( \lambda = 0 \) the differential equation becomes \( y'' = 0 \) with solution \( y = c_1 x + c_2 \), where \( c_1 \) and \( c_2 \) are constants of integration.

Applying the boundary conditions \( y(0) = 0 \) and \( y(l) = 0 \) gives, in turn, \( c_2 = 0 \) and \( c_1 = 0 \). Hence, for \( \lambda = 0 \) the only solution is the trivial solution \( y = 0 \).

**CASE II** When \( \lambda < 0 \) we let \( \lambda = -\mu^2 \). Thus the differential equation becomes \( y'' - \mu^2 y = 0 \) with solution \( y = c_1 \cosh \mu x + c_2 \sinh \mu x \), where \( c_1 \) and \( c_2 \) are constants of integration.

Applying the boundary condition \( y(0) = 0 \) gives \( c_1 = 0 \) and so \( y = c_2 \sinh \mu x \). The second boundary condition \( y(l) = 0 \) yields that

\[
c_2 \sinh \mu l = 0.
\]

Since, \( \sinh \mu l \neq 0 \) then we must have \( c_2 = 0 \). That is, \( y = 0 \): the trivial solution.

**CASE III** When \( \lambda > 0 \) we let \( \lambda = \mu^2 \). Thus the differential equation becomes \( y'' + \mu^2 y = 0 \) with solution \( y = c_1 \cos \mu x + c_2 \sin \mu x \), where \( c_1 \) and \( c_2 \) are constants of integration.

Applying the boundary condition \( y(0) = 0 \) gives \( c_1 = 0 \) and so \( y = c_2 \sin \mu x \) but the second boundary condition \( y(l) = 0 \) yields that

\[
c_2 \sin \mu l = 0.
\]

If \( c_2 = 0 \) then \( y = 0 \): the trivial solution. However, if \( c_2 \neq 0 \) then

\[
\sin \mu l = 0
\]

satisfies the second boundary condition. This equation is sometimes called the characteristic equation and implies that

\[
\mu l = n\pi \quad \text{or} \quad \lambda = \frac{n^2 \pi^2}{l^2}, \quad n = 1, 2, 3, \ldots.
\]

where \( \mu = \sqrt{\lambda} \). (The set of values of \( \lambda \) are called the set of eigenvalues for the Eigenvalue Problem).

Therefore, for any real nonzero \( c_2 \), \( y = c_2 \sin \left( \frac{n\pi x}{l} \right) \) or simply \( y = \sin \left( \frac{n\pi x}{l} \right) \) is a solution to the boundary value problem for each \( n \). In other words, for each eigenvalue in the set of eigenvalues there is a corresponding solution to the given boundary value problem. The set of solutions is called the set of eigenfunctions.

The \( n \)th mode of solution or eigenfunction is written as \( y_n = c_n \sin \left( \frac{n\pi x}{l} \right) \) corresponding to the \( n \)th eigenvalue \( \frac{n\pi}{l} \). Hence, the constant is now dependent on \( n \).

The principle of superposition implies that the sum of all the eigenfunctions is also a solution to the linear homogeneous problem. Thus the general solution to the given boundary value problem is a sum of all the modes of solution. That is,

\[
y = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{l} \right).
\]
Note The set of eigenfunctions \( \{ \sin \left( \frac{n\pi x}{l} \right) \} \), \( n = 1, 2, 3, \ldots \), is an orthogonal set on the interval \([0, l]\) and the above series is a Fourier sine series. The co-efficient \( c_n \) is yet to be determined and is usually determined by initial conditions.

Also, the eigenfunction \( y_n = \sin \left( \frac{n\pi x}{l} \right) \) has \( n - 1 \) zeros in the interval \([0, l]\).

**Exercise 7A**

Consider the differential equation

\[
y'' + \lambda y = 0.
\]

For each of the given following set of boundary conditions, find the eigenvalues and corresponding eigenfunctions.

(a) \( y(0) = 0, \quad y(\pi) = 0 \)

(b) \( y'(0) = 0, \quad y'(\pi) = 0 \)

(c) \( y(0) = 0, \quad y'(1) = 0 \)

(d) \( y(0) + y'(0) = 0, \quad y(1) = 0 \)

(e) \( y(0) = 0, \quad y(\pi) + y'(\pi) = 0 \)

### 7.2 PARTIAL DIFFERENTIAL EQUATIONS

#### 7.2.1 Definitions

A partial differential equation (or PDE) is an equation involving one or more partial derivatives of an unknown function of several variables.

The order of a partial differential equation is the highest derivative term of the dependent variable.

The degree of a partial differential equation is the highest power of the dependent variable and its derivatives.

A linear partial differential equation is one where the dependent variable and its derivatives are of first degree. Otherwise, the partial differential equation is nonlinear.

**Examples of Linear Partial Differential Equations**

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + f(x, t)
\]

\[
\frac{\partial^3 u}{\partial x^2 \partial t} + \alpha(x, t) \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x, t)
\]

\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \alpha(x, t) u.
\]

**Examples of Nonlinear Partial Differential Equations**

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + f(x, t) u^2
\]

\[
\frac{\partial^3 u}{\partial x^2 \partial t} + \alpha(x, t) u \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x, t)
\]

\[
\frac{\partial u}{\partial t} = u \frac{\partial^3 u}{\partial x^3} + \alpha(x, t) u.
\]

A homogeneous partial differential equation is an equation whereby a function of the dependent variable and its derivatives is equal to zero.
Examples of Homogeneous Partial Differential Equations

\[ \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} + f(x, t)u = 0 \]
\[ \frac{\partial^3 u}{\partial x^2 \partial t} + \alpha(x, t)u \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \]
\[ \frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^3} = 0. \]

7.2.2 Solving Linear Partial Differential Equations

Various techniques are used to solve partial differential equations. A simple technique for solving linear and homogeneous partial differential equations with corresponding homogeneous boundary conditions is the method of separation of variables. We shall show this method by way of an example.

Example

Consider the homogeneous linear partial differential equation

\[ \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \quad t \geq 0, \quad 0 \leq x \leq l \]

with homogeneous linear boundary conditions

\[ U(0, t) = U(l, t) = 0, \quad t > 0 \]

and a nonhomogeneous initial condition

\[ U(x, 0) = f(x), \quad 0 < x < l. \]

Finding the solution to the given PDE subject to the given conditions is often called the boundary and initial value problem. The method of separation of variables involves seeking particular solutions of this boundary and initial value problem in the form of products of a function of \( x \) and a function of \( t \). That is,

\[ U(x, t) = X(x)T(t). \]

Therefore,

\[ \frac{\partial U}{\partial x} = X'T \]
\[ \frac{\partial U}{\partial t} = XT' \]

and

\[ \frac{\partial^2 U}{\partial x^2} = X''T. \]

Upon substituting \( U(x, t) \) and its derivatives into the partial differential equation, two ordinary differential equations will be formed. That is,

\[ X''T = XT'. \]

Dividing by \( X(x)T(t) \) gives

\[ \frac{X''}{X} = \frac{T'}{T} \quad (= -\lambda). \]
Since the left-hand side of the last equation is independent of $t$ and is equal to the right-hand side, which is independent of $x$, we conclude that both sides of the equation must be a constant. The constant we used here is $-\lambda$. (The minus sign is used purely for convenience).

The boundary conditions become

\[ U(0, t) = X(0)T(t) = 0 \implies X(0) = 0 \text{ for all } t \]

and

\[ U(l, t) = X(l)T(t) = 0 \implies X(l) = 0 \text{ for all } t. \]

The nonhomogeneous initial condition $U(x, 0) = f(x)$ cannot be simplified here. Therefore, this condition is left until the end. We shall discuss this condition later.

We can now form two ordinary differential equations with appropriate conditions, namely,

\[ X'' + \lambda X = 0 \]

where

\[ X(0) = X(l) = 0 \]

and

\[ T' + \lambda T = 0. \]

This last equation has no homogeneous conditions. However, as discussed before, there is one nonhomogeneous condition. This condition will be used later to determine $U(x, t)$.

The equation in $X$ is simply the eigenvalue problem that we developed before in Section 7.1. (Here, $y$ is replaced by $X$). The 3 cases of $\lambda$ should be considered. Thus the solution for the equation involving $X$ is

\[ X_n(x) = c_n \sin \left( \frac{n\pi x}{l} \right) \quad \text{where} \quad \lambda = \frac{n^2\pi^2}{l^2}, \quad n = 1, 2, 3, \ldots. \]

Note:

$X$ is subscripted in $n$ due to the fact that $\lambda$ is dependent on $n$. Usually we write $\lambda_n$ rather $\lambda$ to signify that there is dependence on $n$.

We now solve the $T$ equation. The solution is

\[ T_n = d_n e^{-\lambda_n t} = d_n e^{-\frac{n^2\pi^2}{l^2} t}. \]

Once again the solution $T$ and the constant $d_n$ is subscripted due to $\lambda$.

Thus we have an infinite number of solutions to the given partial differential equation. That is,

\[ U_n(x, t) = X_n(x)T_n(t) = b_n \sin \left( \frac{n\pi x}{l} \right) e^{-\frac{n^2\pi^2}{l^2} t} \]

is a solution to the given partial differential equation. Note ($b_n = c_n d_n$).

Using the principle of superposition we find that the general solution subject to the boundary conditions is

\[ U(x, t) = \sum_{n=1}^{\infty} U_n(x, t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{l} \right) e^{-\frac{n^2\pi^2}{l^2} t} \]

where $a_n$ needs to be determined.
Finally, the coefficient $a_n$ is determined by the given initial condition. That is,

$$U(x,0) = f(x) \implies f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l}\right).$$

This series is simply a Fourier sine series and $a_n$ can be determined in the usual manner. Hence,

$$b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \left(\frac{n\pi}{l} x\right) dx.$$  

**Special Notes:**

The method of separation of variables works only if the partial differential equation and its boundary conditions are linear and homogeneous. When a boundary or initial condition is nonhomogeneous then the condition can only apply to the general solution, namely,

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t).$$

That is, the non-homogeneous boundary conditions can be only applied to the sum of all eigenfunctions.

If we assume that the given nonhomogeneous initial condition was

$$U(x,0) = f(x) = 1 \text{ (say)}$$

where $0 < x < l$ then

$$1 = \sum_{n=1}^{\infty} a_n \sin \left(\frac{n\pi x}{l}\right).$$

The right hand side of this equation is a Fourier sine series. In other words, we are wanting to find the Fourier sine series for the $f(x) = 1$ on the interval $(0,l)$. Here we use the technique of extending the interval backwards to form a new function $g(x)$ so that

$$g(x) = \begin{cases} 
1 & 0 < x < l, \\
-1 & -l < x < 0.
\end{cases}$$

The new function $g(x)$ is an odd function and hence the Fourier series simplifies to a Fourier sine series (see section 6.3). As a result we find,$t$

$$a_n = \frac{2}{l} \int_{0}^{l} \sin \left(\frac{n\pi}{l} x\right) dx$$

$$= -\frac{2}{n\pi} [\cos(n\pi) - 1]$$

$$= -\frac{2}{n\pi} [(-1)^n - 1].$$

The co-efficient $a_n$ can be further simplified by noting that when

(a) $n$ is even $a_n = 0$ or $a_{2n} = 0$, for $n = 1, 2, 3, \ldots.$

(b) $n$ is odd $a_n = \frac{2}{n\pi}$ or $a_{2n-1} = \frac{2}{(2n-1)\pi}$, for $n = 1, 2, 3, \ldots.$
Therefore, the general solution $U(x,t)$ can now be written

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{l}\right) e^{-\frac{(2n-1)^2\pi^2 t}{l^2}}.$$ 

### Exercise 7B

Use the method of separation of variables to solve each of the following boundary value problems.

1. \[\frac{\partial U}{\partial t} = 4 \frac{\partial^2 U}{\partial x^2}\]
   Subject to:
   \[U(0,t) = 0, U(10,t) = 0 \quad \text{and} \quad U(x,0) = 5 \sin 2\pi x\]

2. \[2 \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}\]
   Subject to:
   \[U(0,t) = 0, U(\pi,t) = 0 \quad \text{and} \quad U(x,0) = 2 \sin 3x - 5 \sin 4x\]

3. \[\frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial x^2}\]
   Subject to:
   \[Y(0,t) = 0, Y(20,t) = 0 \quad \text{and} \quad Y(x,0) = 10 \sin \frac{\pi x}{2}, \quad \frac{\partial Y(x,0)}{\partial t} = 0\]

4. \[9 \frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial x^2}\]
   Subject to:
   \[Y(0,t) = 0, Y(\pi,t) = 0 \quad \text{and} \quad Y(x,0) = 2 \sin x - 3 \sin 2x.\]

5. \[\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y}; \quad U(0,y) = e^{2y}.\]

### 7.3 APPLICATIONS

Particular examples of problems that arise in Applied Mathematics, and which can usually be solved in this manner are given below.

#### 7.3.1 One Dimensional Wave Equation

We have an equation governing small transverse vibrations of an elastic string which is stretched to length $l$ and fixed at the end points:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad t \geq 0, \quad 0 \leq x \leq l \quad (1)$$

where $u = u(x,t)$ is the deflection of the string, and $c^2 = \frac{T}{\rho}$, where $\rho$ is the mass per unit length, and $T$ is the tension of the string.

Typical boundary conditions are

$$u(0,t) = u(l,t) = 0 \quad t > 0. \quad (2)$$
Also, initial conditions such as the following could be imposed.

\[
\begin{align*}
  u(x, 0) &= f(x) \\
  u_t(x, 0) &= g(x)
\end{align*}
\]  

0 < x < l. \quad (3)

**Example**

Consider an elastic string whereby the deflection of the string, \( u(x, t) \), is governed by

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad t \geq 0, \quad 0 \leq x \leq l. \quad (4)
\]

The boundary conditions are

\[
 u(0, t) = u(l, t) = 0 \quad t > 0.
\]

(5)

Physically, these boundary conditions imply that the ends of the string are clamped.

The initial conditions are

\[
\begin{align*}
  u(x, 0) &= x \\
  u_t(x, 0) &= 0
\end{align*}
\]

0 < x < l. \quad (6)

The two initial conditions in (6) relate to the initial displacement and the initial velocity of the string, respectively.

We can use the method of separation of variables to find the solution to the partial differential equation subject to the given conditions. That is, we let

\[
u(x, t) = X(x)T(t).
\]

Therefore,

\[
\frac{\partial u}{\partial x} = X'T \quad \frac{\partial u}{\partial t} = XT'
\]

and

\[
\frac{\partial^2 u}{\partial x^2} = X''T \quad \frac{\partial^2 u}{\partial t^2} = XT''.
\]

Upon substituting \( u(x, t) \) and its derivatives into the partial differential equation we form two ordinary differential equations. That is,

\[
X''T = XT''.
\]

Dividing by \( X(x)T(t) \) gives

\[
\frac{X''}{X} = \frac{T''}{T} = -\lambda.
\]

(7)

We know from the given partial differential equation that \( x \) and \( t \) are independent variables. Since the left-hand side of (7) is a function of \( x \) and the right-hand side is a function of \( t \), we conclude that both sides of the equation must be equal to constant. This constant is \( -\lambda \). (The minus sign is used purely for convenience). Hence, we have 2 second order differential equations to solve subject to the given homogeneous boundary and initial conditions. The non-homogeneous condition will have to be used later when \( u(x, t) \) is found.
The homogeneous boundary conditions become
\[ u(0, t) = X(0)T(t) = 0 \quad \implies \quad X(0) = 0 \quad \text{for all } t \]
and
\[ u(l, t) = X(l)T(t) = 0 \quad \implies \quad X(l) = 0 \quad \text{for all } t . \]
The homogeneous initial condition becomes
\[ u_t(x, 0) = X(x)T'(0) = 0 \quad \implies \quad T'(0) = 0 \quad \text{for all } x . \]

Note:
The non-homogeneous initial condition \( u(x, 0) = x \) will not be used until later.
Hence, the two ordinary differential equations subject to the appropriate conditions are
\[ X'' + \lambda X = 0 \quad (8) \]
where
\[ X(0) = X(l) = 0 \]
and
\[ T'' + \lambda T = 0 \quad (9) \]
where
\[ T'(0) = 0 . \]

Once again, (8) is simply the eigenvalue problem that we have developed in Section 7.1 where \( y \) is replaced by \( X \).

Although there are 3 cases of \( \lambda \) that should be considered, we can eliminate some of these because our boundary conditions are homogeneous. Hence, we seek periodic solutions (trigonometric solutions) due to the homogeneous boundary conditions. Physically, this can be represented by the fact that the string is elastic and therefore, its motion will be periodic.

Thus the solution for (8) is
\[ X_n(x) = c_n \sin \left( \frac{n\pi}{l} x \right) \quad \text{where} \quad \lambda = \frac{n^2 \pi^2}{l^2}, \quad n = 1, 2, 3, \ldots . \quad (10) \]

Note:
\( X \) is subscripted in \( n \) due to the fact that \( \lambda \) is dependent on \( n \). Usually we write \( \lambda_n \) rather \( \lambda \) to signify that there is a dependence on \( n \).

Form (9), the solution to \( T \) is
\[ T_n = d_n \cos(\lambda_n t) + e_n \sin(\lambda_n t) = d_n \cos \left( \frac{n\pi}{l} t \right) + e_n \sin \left( \frac{n\pi}{l} t \right) . \quad (11) \]

Note:
Due to \( \lambda \) being dependent on \( n \) then \( T \) and the constants of integration are dependent on \( n \) and therefore are subscripted in \( n \).
Applying the boundary condition (5) on \( T_n \) in (11), we find that \( e_n = 0 \). Hence, (11) becomes

\[ T_n = d_n \cos\left(\frac{n\pi t}{l}\right) \quad n = 1, 2, 3, \ldots \]  

(12)

Thus, multiplying both (10) and (12) together gives us an infinite number of solutions to the given partial differential equation (4) subject to the given homogeneous boundary conditions in (5).

That is,

\[ u_n(x, t) = X_n(x)T_n(t) = a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi t}{l}\right) \]  

(13)

is the \( n \)th mode of solution to the given partial differential equation. Note \( a_n = c_n d_n \).)

Using the principle of superposition and (13), the general solution is

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi t}{l}\right) \]  

(14)

where \( a_n \) needs to be determined.

The coefficient \( a_n \) in (14) is determined by the given nonhomogeneous initial condition found (6). That is,

\[ u(x, 0) = x \quad \Rightarrow \quad x = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{for} \quad 0 < x < l. \]

This series is simply a Fourier sine series and \( a_n \) can be determined in the usual manner. Hence,

\[ a_n = \frac{2}{l} \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx \quad n = 1, 2, 3, \ldots. \]

Integrating by parts, we find that

\[ a_n = \frac{2(-1)^{n+1} l}{n\pi}. \]

Upon substituting \( a_n \) back into (14), then the general solution to PDE in (4) subject to the given conditions in (5) and (6) is

\[ u(x, t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi t}{l}\right). \]

7.3.2 Two Dimensional Wave Equation

The governing equation for the transversal vibration of a membrane is given by

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \]

where \( u(x, y, t) \) is the deflection of the membrane, and \( c^2 = \frac{T}{\rho} \), where \( \rho \) is the mass of the membrane per unit area, and \( T \) is the tension in the membrane.

Solution to the \( 2-D \) wave equation can be obtained by using the separation of variables technique provided the boundary conditions homogeneous.
For typical homogeneous boundary conditions we can use the following rectangular membrane whereby all the boundary conditions are zero.

Here, \( u(x, y, t) = 0 \) on the boundary for all \( t \). That is,

\[
 u(0, y, t) = 0 = u(a, y, t)
\]

and

\[
 u(x, 0, t) = 0 = u(x, b, t).
\]

The initial conditions could be:

\[
 u(x, y, 0) = f(x, y)
 u_t(x, y, 0) = g(x, y)
\]

Since the boundary conditions are homogeneous then we can use the separation of variables technique. Now \( u \) is dependent on \( x, y \) and \( t \) therefore we assume that the solution has the form

\[
 u(x, y, t) = X(x)Y(y)T(t).
\]

This form of solution is substituted into the given pde. As a result, we have

\[
 X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = \frac{1}{c^2} X(x)Y(y)T''(t).
\]

Dividing by \( X(x)Y(y)T(t) \) we have

\[
 \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{c^2} \frac{T''(t)}{T(t)} = -\lambda.
\]

Using the fact that \( x, y \) and \( t \) are independent variables, the above equation reduces to 3 ordinary differential equations that can be solved by methods that have been previously discussed.

Note: The \( n \)-dimensional wave equation is

\[
 \nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]

where

\[
 \nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}
\]

and

\[
 u = u(x_1, \ldots, x_n, t).
\]
7.3.3 The Heat Equation

The heat flow in a body of homogeneous material is governed by the heat equation

\[ \nabla^2 u(x, y, z, t) = \frac{1}{c^2} \frac{\partial u(x, y, z, t)}{\partial t} \]

where \( u(x, y, z, t) \) is the temperature at the point \( (x, y, z) \) in the body at time \( t \), \( c^2 = \frac{K}{\rho \sigma} \) is the diffusivity of the body, and \( K \) is the thermal conductivity, \( \sigma \) is specific heat and \( \rho \) is the density of material.

Often we consider the temperature in a uniform bar of length \( l \), which is oriented along the \( x \)-axis, and where both ends of the bar are held at zero temperature. We obtain the equation

\[ \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial u(x, t)}{\partial t} \]

and

\[ u(0, t) = u(l, t) = 0 \]

where the initial temperature can be given by

\[ u(x, 0) = f(x) \]

This PDE is called the 1-D heat equation.

**Example**

Consider the boundary and initial value problem

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad t \geq 0, \quad 0 \leq x \leq 1 \]

with homogeneous linear boundary conditions

\[ u_x(0, t) = u_x(l, t) = 0, \quad t > 0 \]

and nonhomogeneous initial condition

\[ u(x, 0) = x^2, \quad 0 < x < 1. \]

Let

\[ u(x, t) = X(x)T(t) \]

then

\[ \frac{\partial u}{\partial x} = X'T \quad \quad \frac{\partial u}{\partial t} = XT' \]

and

\[ \frac{\partial^2 u}{\partial x^2} = X''T. \]
Upon substituting $u(x,t)$ and its derivatives into the partial differential equation, two ordinary differential equations will be formed. That is, 

$$X''T = XT'.$$

Dividing by $X(x)T(t)$ gives 

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$ 

We know that $x$ and $t$ are independent variables. However, in the above equation, we have the left-hand side being a function of $x$ and the right-hand side a function of $t$. Therefore, it is concluded that both sides of the equation must be a constant. The constant we used here is $-\lambda$. (The minus sign is used purely for convenience).

The homogeneous boundary conditions become 

$$u_x(0,t) = X'(0)T(t) = 0 \implies X'(0) = 0 \text{ for all } t$$

and 

$$u_x(1,t) = X'(1)T(t) = 0 \implies X'(1) = 0 \text{ for all } t.$$ 

The nonhomogeneous initial condition $u(x,0) = x^2$ cannot be simplified here. Therefore, this condition is left until we find the $n$th mode of solution.

We can now form two ordinary differential equations with appropriate conditions, namely, 

$$X'' + \lambda X = 0$$

where 

$$X'(0) = X'(1) = 0$$

and 

$$T' + \lambda T = 0.$$ 

This last equation has a nonhomogeneous condition and therefore cannot be applied separately to $T$.

The equation in $X$ is simply the eigenvalue problem that we developed before in section 7.1. (Here, $y$ is replaced by $X$). The 3 cases of $\lambda$ should be considered, however, due the homogeneous boundary conditions we seek trigonometric solutions.

(Note: The boundary conditions are slightly different in this problem.)

Let $\lambda = \mu^2$. Thus the solution for $X(x)$ is 

$$X = a \cos \mu x + b \sin \mu x.$$ 

The co-efficients, $a$ and $b$ need to be determined. From the given boundary conditions it can be seen that we require 

$$X'(x) = -a\mu \sin \mu x + b\mu \cos \mu x.$$ 

Using $X'(0) = 0$ we find that $b = 0$ here. Therefore, 

$$X(x) = a \cos \mu x.$$
Using $X'(1) = 0$, we find that
$$-a\mu \sin \mu = 0.$$ 
This equation implies that either $a = 0$ or $\sin \mu = 0$. However, if $a = 0$ then $X(x) = 0$ and therefore, we obtain the trivial solution. Hence, for non-trivial solutions
$$\sin \mu = 0.$$ 
This implies
$$\mu = n\pi \quad \text{for} \quad n = 0, 1, 2, \ldots.$$ 
and therefore, $\lambda = n^2\pi^2$ for $n = 0, 1, 2, \ldots$. 

*Note:* Here $n = 0$ is included as its value does not produce the trivial solution. However, $n = 0 \implies \lambda = 0$. The $\lambda = 0$ case should have been considered separately but it will be included as part of the trigonometric solution here! It will turn out that the solution for $\lambda = 0$ case is the $a_0$ term in a Fourier series.

Now $X(x)$ becomes
$$X_n(x) = a_n \cos(n\pi x) \quad \text{where} \quad \lambda = n^2\pi^2, \quad n = 1, 2, 3, \ldots.$$ 

We now solve the $T$ equation. The solution is
$$T_n = d_n e^{-\lambda_n t} = d_n e^{-n^2\pi^2 t}.$$ 

Once again the solution $T$ and the constant $d_n$ is subscripted due to $\lambda$ being dependent on $n$.

Thus we have an infinite number of solutions to the given partial differential equation. That is,
$$u_n(x, t) = X_n(x)T_n(t) = c_n \cos(n\pi x) e^{-n^2\pi^2 t}$$ 
is a solution to the given partial differential equation. (*Note $c_n = a_n d_n$.*)

Using the principle of superposition we find that the general solution subject to the boundary conditions is
$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} a_n \cos(n\pi x) e^{-n^2\pi^2 t}$$
where $a_n$ needs to be determined.

Finally, the coefficient $a_n$ is determined by the given initial condition. That is,
$$u(x, 0) = x^2 \implies x^2 = \sum_{n=1}^{\infty} a_n \cos(n\pi x), \quad 0 < x < 1.$$ 

This series is simply a Fourier cosine series and $a_n$ can be determined in the usual manner. Hence,
$$a_n = 2 \int_0^1 x^2 \cos(n\pi x) \, dx \quad n = 0, 1, 2, \ldots.$$
Intergrating by parts we obtain
\[ a_n = \frac{4(-1)^n}{n^2\pi^2} \quad \text{for} \quad n \neq 0. \]

When \( n = 0 \) it is found that
\[ a_0 = \frac{2}{3}. \]

Thus the general solution of the given PDE is
\[ u(x, t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x) e^{-n^2\pi^2 t}. \]

Note:
Recall that the \( a_0 \) is divided by 2 in a Fourier series. This is due to an orthogonality condition for \( n=0 \) (see section 6.1.4).

### 7.3.4 Steady State Temperatures

Steady state temperatures are ones which are independent of time. Thus the resulting partial differential equation becomes
\[ \nabla^2 u(x, y, z) = 0. \]

This equation is also used to describe electric (or gravitational) potential effect due to a charge (or mass) distribution at points where there is no charge (or mass).

### 7.3.5 Angular displacements in a Shaft

Let \( \theta(x, t) \) be angular displacement (or twist) of a vibrating shaft of unit length 1 and a circular cross section with axis along the \( t \)-axis.

The displacement due to an initial twist of \( \theta(x, 0) = f(x) \) is given by
\[ \frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2} \]

where \( a \) is a constant. The angular displacement \( \theta(x, t) \) is subject to the boundary conditions
\[ \theta(0, t) = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial x} \bigg|_{x=1} = 0, \quad t > 0 \]

and initial conditions
\[ \theta(x, 0) = x \quad \text{and} \quad \frac{\partial \theta}{\partial t} \bigg|_{t=0} = 0, \quad 0 < x < 1. \]
Solving for $\theta(x,t)$, we let $\theta(x,t) = X(x)T(t)$ upon substituting into the partial differential equation we find

$$X'' + \lambda^2 X = 0$$

and

$$T'' + \lambda^2 T = 0.$$ 

Therefore,

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

and

$$T(t) = c_3 \cos a \lambda t + c_4 \sin a \lambda t.$$ 

The boundary conditions $X(0) = 0$ and $X'(1) = 0$ give $c_1 = 0$ and $c_2 \cos \lambda = 0$, respectively.

For non-trivial solutions we require that $\cos \lambda = 0$. That is,

$$\lambda = \frac{(2n - 1)\pi}{2}, \quad n = 1, 2, 3, \ldots.$$ 

Also, the initial condition $T'(0) = 0$ gives $c_4 = 0$ and hence,

$$\theta_n(x,t) = X(x)T(t) = A_n \sin \left( \frac{(2n - 1)\pi x}{2} \right) \cos \left( a \frac{(2n - 1)\pi t}{2} \right).$$ 

Using the final condition, that is, $t = 0$ we have

$$\theta(x,0) = x = \sum_{n=1}^{\infty} A_n \sin \left( \frac{(2n - 1)\pi x}{2} \right).$$ 

The term on the right hand side is a Fourier sine series. Therefore, $A_n$ is able to be determined by finding $f(x) = x$ as a Fourier sine series using the methods discussed in Chapter 6.

Hence,

$$A_n = \frac{8(-1)^{n+1}}{(2n - 1)^2 \pi^2}.$$ 

Hence, the twist angle of the shaft is

$$\theta(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n - 1)^2} \sin \left( \frac{(2n - 1)\pi x}{2} \right) \cos \left( a \frac{(2n - 1)\pi t}{2} \right).$$

7.3.6 Other Related Equations

Other instances in applied mathematics where similar equations arise are:

(a) electric potential between parallel plates, and

(b) potential in a quadrant.
Exercise 7C

Use separation of variable to solve the following partial differential equations subject to the given initial and boundary conditions.

1. (a) \[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 \; ; \; 0 < t < \infty \]
   where \[ u(0, t) = 0 \] \[ u(1, t) = 0 \] for \( 0 < t < \infty \)
   and \[ u(x, 0) = x(1 - x) \].

(b) \[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 , \]
   where \[ u(0, t) = u(1, t) = 0 , \; t > 0 \] and \[ u(x, 0) = 1 , \; 0 < x < 1 . \]

(c) \[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 , \]
   where \[ \frac{\partial u}{\partial x} = 0 , \; \text{for} \; x = 0 , \; t > 0 , \]
   and \[ \frac{\partial u}{\partial x} = 0 , \; \text{for} \; x = 1 , \; t > 0 . \]
The initial condition being \[ u(x, 0) = x , \; 0 < x < 1 . \]

2. (a) \[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 \; ; \; 0 < t < \infty \]
   where \[ u(0, t) = 0 \] \[ u(1, t) = 0 \] for \( 0 < t < \infty \)
   with \[ u(x, 0) = 0 \] \[ u_t(x, 0) = 1 \] for \( 0 < x < 1 . \)

(b) \[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \pi \; ; \; 0 < t < \]
   where \[ u(0, t) = 0 \] \[ u(\pi, t) = 0 \] for \( 0 < t < \infty \)
   with \[ u(x, 0) = x \] \[ u_t(x, 0) = 0 \] for \( 0 < x < \pi . \)

3. Find the steady state temperature (ie. independent of time) in a two-dimensional semi-infinite slab of width \( a \) where the temperatures on the 3 boundaries are given by

\[ u(0, y) = 0 , \; u(a, y) = 0 \]

and

\[ u(x, 0) = x(a - x) . \]

You are also given that \[ u(x, y) \to 0 \; \text{as} \; y \to \infty . \]

4. Use the method of separation of variables to find a solution to the problem for each given value of \( f(x) \).

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} , \; 0 < x < a , \; 0 < t < \infty , \]

subject to

\[ u(0, t) = 0 \]
\[ u(a, t) = 0 \]
\[ 0 < t < \infty , \]

\[ \frac{\partial u}{\partial t} = 0 , \; \text{when} \; t = 0 , \] for \( 0 < x < a . \)

where

(i) \[ f(x) = x(a - x) , \] and

(ii) \[ f(x) = \begin{cases} x , & 0 < x < \frac{1}{2}a , \\ a - x , & \frac{1}{2}a < x < a. \end{cases} \]

5. Show that the solution to the one dimensional wave equation with the usual boundary conditions, but with initial conditions

\[ u(x, 0) = f(x) = \begin{cases} \frac{2kx}{l} , & 0 < x < \frac{l}{2} , \\ \frac{2k}{l^2}(l - x) , & \frac{l}{2} < x < l \end{cases} \]

and

\[ \frac{\partial u}{\partial t} \bigg|_{t=0} = g(x) = 0 \]

is

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{(-1)^n 8k}{(2n - 1)^2 \pi^2} \times \left( \sin \left( \frac{(2n - 1)\pi x}{l} \right) \cos \left( \frac{(2n - 1)\pi t}{l} \right) \right) . \]

i.e. Show \( F_n = 0 \) and \( E_n = \frac{8k}{n^2 \pi^2} \sin \frac{n\pi}{l} \).
This problem corresponds to an initial triangular deflection below.

Consider the temperature in a uniform bar of length $l$ which is oriented along the $x$-axis. Both ends of the bar are held at zero temperature. If the initial temperature in the bar is

$$f(x) = \begin{cases} x, & 0 < x < \frac{l}{2} \\ l - x, & \frac{l}{2} < x < l \end{cases}$$

where $x$ is the distance measured from one end, show that the temperature distribution in the bar at time $t$ is given by

$$u(x, t) = \sum_{n=0}^{\infty} b_n \exp \left( -\left\lfloor \frac{c_n \pi^2}{l^2} \right\rfloor t \right) \sin \frac{n\pi x}{l}$$

where

$$b_n = \begin{cases} 0, & n \text{ even} \\ \frac{4l}{n^2 \pi^2}, & n = 1, 5, 9, \ldots \\ \frac{-4l}{n^2 \pi^2}, & n = 3, 7, 11, \ldots \end{cases}$$

Find the steady state temperature (ie. independent of time) in a two-dimensional semi-infinite slab of width $a$ where the temperatures on the 3 boundaries are given by

$$u(0, y) = 0, \quad u(a, y) = 0$$

and

$$u(x, 0) = x(a - x).$$

You are also given that $u(x, y) \to 0$ as $y \to \infty$. 

Consider a thin metal bar of homogeneous material and of length $L$ which is placed on the $x$-axis of a $(x, y)$ coordinate system.

The bar is uniformly heated. Suppose that the cross-sectional dimensions are relatively small and that the temperature $U$ can be considered constant on any given cross-section. Thus, the temperature $U$ is a function of $x$ and $t$ only. Suppose also that the bar experiences radiation of temperature $U_0$ from the surroundings, then the variation in temperature in the bar is governed by the one-dimensional heat flow or heat conduction equation:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - \lambda^2(U - U_0)$$

where $\lambda^2 = \frac{hP}{k} A_c$ is a constant. (Note: $h$ is the heat transfer co-efficient on the outside of the bar, $P$ and $A_c$ are respectively, the perimeter and the cross-sectional area of the bar and $k$ is the thermal conductivity of the bar).

(1) Suppose that the temperature in the metal bar has reached a steady-state temperature, that is $\frac{\partial U}{\partial t} = 0$. Hence, $U$ is a function of $x$ only ( i.e $U = U(x)$).

(a) Show that by letting $\Theta(x) = U(x) - U_0$ that the heat equation reduces to a second order differential equation of the form $\frac{d^2 \Theta}{dx^2} - \lambda^2 \Theta = 0$.

(b) Solve the differential equation in 1(a). Check your solution using Mathematica.

(c) At the end of the bar at $x = 0$ the temperature of the bar $\Theta = U(0) - U_0 = \Theta_b$, and at the end of the bar $x = L$ there is convection so that $\frac{dU}{dx} + \gamma(U - U_0) = 0$. That is, $\frac{d\Theta}{dx} + \gamma \Theta = 0$ at $x = L$ where $g = \frac{k}{h}$ is a constant. Find $\Theta(x)$ using these boundary conditions and check your solution using Mathematica.

(d) Use Mathematica

(i) and assuming $\Theta_b = 1$, $L = 1$, $\gamma = 1$ and $\lambda = 1$, plot $\Theta(x)$ for $x$ from 0 to 1.

(ii) By varying $\gamma$, $\lambda$ and $\Theta_b$, investigate the behaviour of the metal bar temperature through plots of $\Theta(x)$ for $x$ from 0 to 1.
(e) Suppose that at \( x = 0 \) the boundary condition does not change. However, at \( x = L \) the bar is insulated ie \( \frac{d\Theta}{dx} = 0 \).

(i) Determine the temperature \( \Theta(x) \) using these conditions. Check your solution using Mathematica.

(ii) Assuming \( \Theta_b = 1 \), \( L = 1 \), \( g = 1 \) and \( a = 1 \), plot \( \Theta(x) \) for \( x \) from 0 to 1.

(iii) With the aid of plots, or otherwise, discuss the effect of varying the parameters \( \gamma \), \( \lambda \) and \( \Theta_b \).

(iv) Discuss the difference between the change of boundary conditions on the metal bar temperature.

(f) (i) Determine the solution of (e) if \( L \) is allowed to become large (ie \( L \to \infty \)).

(ii) Compare your result to a Mathematica to plot of \( \Theta(x) \) using the values in 1f(i).

(2) Consider the one dimensional (1-D) heat flow equation with the assumption that the metal bar is insulated from its surroundings. Hence, the 1-D equation reduces to

\[
\frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2},
\]

where \( k \) is called the diffusivity of the metal bar.

(a) Using the separation of variables technique, show that

\[
U(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\lambda_n t)e^{-\lambda_n kt} + \sum_{n=1}^{\infty} b_n \sin(\lambda_n t)e^{-\lambda_n kt},
\]

where \( \lambda_n = \frac{n\pi}{L} \).

(b) Apply the following temperature conditions for the bar:

At the boundaries: \( U(0, t) = 0 \) and \( U(L, t) = 0 \) and Initially: \( U(x, 0) = f(x) \).

Show that \( a_n = 0 \), and \( b_n = \frac{2}{L} \int_0^L f(x) \sin(\lambda_n x) \, dx \) for \( n = 0, 1, 2, \ldots \).

(c) Let \( f(x) = x \) and \( L = 1 \). Use Mathematica to

(i) Evaluate \( b_n \),

(ii) Determine the first 5 coefficients in the series,

(iii) Sum the first 5 terms in the series. Graph the sum for \( t = 0 \) to \( t = 1 \) in steps of 0.2.

(iv) Repeat c(i)- c(iii) and compare results to those found in 2c(iii) where

\[
f(x) = \begin{cases} 
  x & 0 < x < 0.5 \\
  1 - x & 0.5 < x < 1 
\end{cases}
\]

(3) Using the results above, write a 500 word report to discuss heat flow in a metal bar. You report should include at least one application of where heat flow is important.
Chapter 8: Numerical Methods 1

8.1 INTRODUCTION

In mathematics we strive for exactness as approximations leads to inaccuracies and errors that cause untold problems. For example, many of the differential equations that we encounter in mathematics cannot be solved exactly and therefore, we have to find approximate their solution. This leads to errors. One way to approximate the solution to a given equation is called Numerical Methods. In the Numerical Methods section of MATH202 we shall be looking at the most efficient way(s) of approximating the solution to a given differential equation.

8.1.1 Significant Figures Rules

(1) Coming from the left, the first non-zero digit is the first significant figure.

(2) All figures following the first figure are also significant unless the number is a whole number ending in zero.

(3) Final zeros in a whole number may or may not be significant.

Example

213 has 3 significant figures.

0.0213 has 3 significant figures.

21.3 has 3 significant figures.

21300 may have 3, 4 or 5 significant figures.

0.02130 has 4 significant figures.

8.1.2 Error Definitions

\[
\text{Absolute Error} = | \text{True Value} - \text{Approx Value} |.
\]

\[
\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True Value}|}.
\]

\[
\text{Relative % Error} = \frac{\text{Absolute Error}}{|\text{True Value}|} \times \frac{100}{1}.
\]
As measure of accuracy the relative error is more meaningful than the absolute error.

**Example**

The distance from Sydney to Melbourne is 877 kms. Suppose we measured the distance to be 900 kms.

\[
\text{Absolute Error} = |877 - 900| = 23 \text{ kms}.
\]

\[
\text{Relative Error} = \frac{23}{877} = 2.62 \times 10^{-2}.
\]

### 8.1.3 Stepsize

Let \( h \) represent the positive difference between consecutive values along, say, the \( x \) axis. The difference of consecutive \( x \) values is usually ‘reasonably’ small and equally spaced. Thus we can form a sequence of \( x \)-values, \( x_1, x_2, x_3, \ldots \), where

\[
x_1 = x_0 + h
\]
\[
x_2 = x_1 + h = x_0 + 2h
\]
\[
x_3 = x_2 + h = x_0 + 3h
\]

\[ \vdots \]
\[
x_n = x_{n-1} + h = x_0 + nh
\]

In numerical analysis, \( h \) is usually called the **step size** or **interval length**. For instance, suppose that \( x \in [0, 1] \) and the interval is divided into 10 intervals then \( h = \frac{1 - 0}{10} = 0.1 \).

Generally, if \( x \in [a, b] \) and the interval is divided into \( n \) intervals then

\[
h = \frac{b - a}{n}.
\]

**Note:** We have used \( x \) here but we can extend the step size definition to any variable that requires increments.

### 8.2 APPROXIMATING FUNCTIONS

#### 8.2.1 Taylor Polynomials

Values of polynomials, such as \( p(x) = x^3 + 2x^2 - 3x + 1 \), may be readily calculated for any value of \( x \) whereas many other functions, such as \( f(x) = \sin x \), cannot be evaluated, for most values of \( x \), without the aid of a calculator. In this section we show how to construct a polynomial of a given degree that approximates a given function \( f \) and its derivatives, near a given point \( a \).
Definition

Suppose $f$ is a given function which is $n$ times differentiable at a given point $x = a$. The $n$th Taylor polynomial for $f$ about $a$ is the polynomial, $p_n$, of degree $n$ which agrees with $f$ and its first $n$ derivatives at the point $x = a$.

That is, we require that:

$$p_n(a) = f(a); \quad p'_n(a) = f'(a); \quad p''_n(a) = f''(a); \quad \ldots; \quad p^{(n)}_n(a) = f^{(n)}(a).$$

Notation:

For $k = 0, 1, 2, \ldots$ and any function $f$,

$$f^{(k)}(x)$$

denotes the $k$th derivative of the function $f$ evaluated at the point $x$.

That is

$$f^{(k)}(x) = \frac{d^k f}{dx^k}(x).$$

For the special case when $k = 0$, $f^{(0)}(x) = f(x)$.

Any polynomial of degree $n$ can be written in the form

$$p_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \ldots + a_n(x-a)^n$$

with suitably chosen numbers (called coefficients) $a_0, a_1, a_2, \ldots, a_n$. If $p_n$ is the Taylor polynomial for $f$ then it can be shown (by differentiating repeatedly and then evaluating at $x = a$, see Exercise 9A Q5) that

$$a_k = \frac{1}{k!} f^{(k)}(a) \quad \text{for} \quad k = 0, 1, 2, \ldots, n.$$  

Thus the Taylor polynomial of degree $n$ for $f$ about the point $x = a$, is given by

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$ 

Example:

Suppose $f(x) = \ln(x)$ and $a = 1$.

<table>
<thead>
<tr>
<th>$f(1)$</th>
<th>$ln(1) = 0$, and so $p_0(x) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x) = x^{-1}$</td>
<td>$f'(1) = 1$; $p_1(x) = 0 + (x-1) = (x-1)$</td>
</tr>
<tr>
<td>$f''(x) = -x^{-2}$</td>
<td>$f''(1) = -1$; $p_2(x) = (x-1) - \frac{1}{2!}(x-1)^2 = (x-1) - \frac{(x-1)^2}{2}$</td>
</tr>
<tr>
<td>$f'''(x) = 2x^{-3}$</td>
<td>$f'''(1) = 2$; $p_3(x) = (x-1) - \frac{2}{3!}(x-1)^3 = (x-1) - \frac{(x-1)^3}{3}$</td>
</tr>
<tr>
<td>$f^{(4)}(x) = -(3)(2)x^{-4}$</td>
<td>$f^{(4)}(1) = -(3)(2)$; $p_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-4)^4}{4}$</td>
</tr>
</tbody>
</table>
In general, 
\[ f^{(k)}(x) = (-1)^{k+1}(k-1)!x^{-k} \]
\[ p_k(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \ldots + \frac{(-1)^{k+1}(x-1)^k}{k} \]
\[ = \sum_{n=1}^{k} (-1)^{n+1}(x-1)^n \]

The following graph shows a plot of the function, \( \ln(x) \), together with plots of the first few Taylor polynomials. \( p_0(x) \) coincides with the \( x \)-axis and so is not shown.

From the graph it can be seen that the polynomials, particularly \( p_3 \), approximate \( \ln(x) \) for values of \( x \) close to 1. The approximation is not so good further away from 1.

If we attempt to approximate \( \ln(1.1) \) then we can evaluate the polynomials at the point \( x = 1.1 \). The values (correct to 6 decimal places) obtained are

\[ p_0(1.1) = 0 \quad p_1(1.1) = 0.1 \quad p_2(1.1) = 0.095 \]
\[ p_3(1.1) = 0.095333 \quad p_4(1.1) = 0.095308 \]

The correct value for \( \ln(1.1) \), correct to 6 decimal places, is 0.095310.

Example:

Let \( f(x) = \sin x \) and \( a = \frac{\pi}{2} \).

Then
\[ p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \]
with \( f(x) = \sin x \) and \( a = \frac{\pi}{2} \).

We are required to calculate the terms \( \frac{f^{(k)}(\frac{\pi}{2})}{k!} \) in the series; hence

\[ f(x) = \sin x; \quad f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1 \]
\[ f'(x) = \cos x; \quad f'\left(\frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0 \]
\[ f''(x) = -\sin x; \quad f''\left(\frac{\pi}{2}\right) = -\sin \frac{\pi}{2} = -1 \]
\[ f'''(x) = -\cos x; \quad f'''\left(\frac{\pi}{2}\right) = -\cos \frac{\pi}{2} = 0 \]
\[ f^{(4)}(x) = \sin x; \quad f^{(4)}\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1, \quad \text{etc.} \]

Since \( f^{(4)} = f \), further values in the table will simply repeat.

It is obvious that all odd order derivatives are zero at \( x = \frac{\pi}{2} \), so \( f^{(2k-1)}\left(\frac{\pi}{2}\right) = 0 \).
Upon substitution, for $n$ even, we get

$$p_n(x) = 1 - \frac{1}{2!} (x - \frac{\pi}{2})^2 + \frac{1}{4!} (x - \frac{\pi}{2})^4 + \cdots (-1)^{n/2} \frac{1}{n!} (x - \frac{\pi}{2})^n$$

$$= \sum_{k=0}^{n/2} (-1)^k \frac{(x - \frac{\pi}{2})^{2k}}{(2k)!}.$$ 

In particular

$$p_0(x) = 1,$$
$$p_1(x) = p_2(x) = 1 - \frac{1}{2!} (x - \frac{\pi}{2})^2,$$
$$p_3(x) = p_4(x) = 1 - \frac{1}{2!} (x - \frac{\pi}{2})^2 + \frac{1}{4!} (x - \frac{\pi}{2})^4.$$ 

The graph below shows this function and the polynomials.

8.2.2 Taylor Series Expansions

If the function $f$ can be differentiated infinitely often at a point $x = a$ then the Taylor polynomials for $f$ of all degrees can be found. It is normal to consider all these polynomials at once by combining them into an infinite series.

**Definition:**

Suppose $f$ is a given function which is infinitely differentiable at a given point $x = a$. The **Taylor series expansion** for $f$ about the point $x = a$ is the infinite series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$ 

The point $x = a$ is called the **centre** of the expansion.

In the special case where the given point is $a = 0$ the series is called the **Maclaurin series expansion** for $f$. This simplifies to

$$f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$ 

8.3 TABLE OF STANDARD (MACLAURIN) SERIES

A requirement for success is that you are familiar with the following Table of Standard Series.

<table>
<thead>
<tr>
<th>Function</th>
<th>Maclaurin Series</th>
<th>Region of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1-x} )</td>
<td>( 1 + x + x^2 + x^3 + \ldots + x^n + \ldots ) Geometric Series</td>
<td>(</td>
</tr>
<tr>
<td>((1 + x)^p)</td>
<td>( 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \ldots )</td>
<td>All ( x ) when ( p \in \mathbb{N} ), (</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots ) Binomial Series</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \ln(1+x) )</td>
<td>( x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n+1} \frac{x^n}{n} + \ldots )</td>
<td>(</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots )</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots )</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \sinh x )</td>
<td>( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots )</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \cosh x )</td>
<td>( 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots )</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \tan^{-1} x )</td>
<td>( x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots ) Gregory’s Series</td>
<td>(</td>
</tr>
<tr>
<td>( \tanh^{-1} x )</td>
<td>( x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n+1}}{2n+1} + \ldots )</td>
<td>(</td>
</tr>
</tbody>
</table>

8.4 ERROR IN APPROXIMATION

8.4.1 The Remainder for Taylor Series

From the examples and graphs above it can be seen that the Taylor polynomials approximate the given function quite well for values of \( x \) close to the centre of expansion \( a \). Further, as the degree of the polynomial is increased the polynomials become even better approximations to the function.

Example:
Suppose \( f(x) = \frac{1}{1-x} \) and centre \( a = 0 \).

The Maclaurin polynomials for \( f \) are

\[ p_n(x) = 1 + x + x^2 + x^3 + \ldots + x^n \]
and they can be used to attempt to find approximate values for \( f(x) \).

If \( x = 0.5 \) then \( f(x) = 2 \) and if \( x = 2 \) then \( f(x) = -1 \). The values for \( p_n(0.5) \) and \( p_n(2) \), correct to 4 decimal places, are

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p_n(0.5) )</th>
<th>( p_n(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1.75</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>1.875</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>1.9375</td>
<td>31</td>
</tr>
<tr>
<td>5</td>
<td>1.9688</td>
<td>63</td>
</tr>
<tr>
<td>6</td>
<td>1.9844</td>
<td>127</td>
</tr>
<tr>
<td>7</td>
<td>1.9922</td>
<td>255</td>
</tr>
<tr>
<td>8</td>
<td>1.9961</td>
<td>511</td>
</tr>
<tr>
<td>9</td>
<td>1.9980</td>
<td>1023</td>
</tr>
<tr>
<td>10</td>
<td>1.9990</td>
<td>2047</td>
</tr>
</tbody>
</table>

From this table it can be seen that the values of the polynomials at 0.5 approach the value of the function as the degree is increased, but, unlike our previous examples, the values at 2 are not close to the function value and they get worse as the degree is increased.

In the next section we will see how to determine for what values of \( x \) the polynomials will give good approximations to the function and, even how to find out how close the approximations really are.

**Definition:**

Suppose \( f \) is a given function and \( p_n \) is the the Taylor polynomial for \( f \) of degree \( n \) with centre \( x = a \). Then we define the \( n \)th Taylor Remainder, \( R_n \), of \( f \), by

\[
R_n(x) = f(x) - p_n(x) = f(x) - \left[ f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right].
\]

The error in approximating \( f(x) \) by \( p_n(x) \) is given by \(|R_n(x)|\).

Taylor Remainders are related to the remainders discussed in Section 13.3.5. However they are not precisely the same and in particular, for the Taylor Remainder, \( n \) refers to the degree of the polynomial and not to the number of terms in the expansion.

### 8.4.2 Taylor’s Theorem

If we are using Taylor polynomials to find approximate values for a function \( f \) then we can’t use the formula above to calculate \( R_n(x) \) (since we don’t know the value of \( f(x) \)). However there are some other formulae for \( R_n \) which won’t allow us to find it precisely but will allow us to estimate its size.
Theorem.

Suppose \( f \) is \((n + 1)\) times differentiable on an interval containing \( a \) and \( x \) and \( R_n \) is the \( n \)th Taylor Remainder for \( f \) with centre \( x = a \). Then there is a point \( c \), lying between \( a \) and \( x \), such that

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.
\]

The point \( c \) in this theorem can not usually be determined. It will change if we consider a different point \( x \) (or a different degree, \( n \), or a different centre, \( a \), or a different function, \( f \)).

The formula given for \( R_n(x) \) in the Theorem is known as the Lagrange Form for the remainder. There are two other formulae for \( R_n \):

The Integral Form of the Remainder

\[
R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt,
\]

and Cauchy’s Form of the Remainder

\[
R_n(x) = \frac{(x-a)(x-d)^n}{n!} f^{(n+1)}(d),
\]

where \( d \) is some number between \( a \) and \( x \).

Example:
Consider \( f(x) = \sin x \).
Using the first three non-zero terms of the Maclaurin series for \( \sin x \), calculate an approximation to \( \sin 1 \), and estimate the error.

Method

The Maclaurin series for \( f \) is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots
\]

The first 3 non-zero terms in this series give \( x - \frac{x^3}{3!} + \frac{x^5}{5!} \) which is the 5th Maclaurin polynomial for \( \sin(x) \). In fact, since the coefficient for \( x^6 \) in the series is 0, this is also the 6th Maclaurin polynomial, \( p_6(x) \). To compute \( \sin 1 \), we have

\[
\sin 1 \approx p_6(1) = 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120}.
\]

Using the Lagrange form of the remainder, we have

\[
R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)
\]

where \( c \) is some number between \( a \) and \( x \). On substituting \( x = 1 \), \( a = 0 \) and \( n = 6 \), we get

\[
|R_6(1)| = \left| \frac{1}{7!} f^{(7)}(c) \right| = \frac{1}{7!} | \cos c | \leq \frac{1}{7!} = \frac{1}{5040}.
\]
To an accuracy of 4 significant figures, \( \frac{1}{5040} = 1.984 \times 10^{-4} \).

On the other hand, using a calculator, \(|\sin 1 - p_0(1)| = 1.957 \times 10^{-4}\) so that our error estimate is a little large but quite close to the correct value.

Sometimes, these formulae can be quite messy in application, and other simpler error estimates may be sought. In the above example, where the series is a strictly alternating series, a method given in the alternating series section of Sequences and Series will be much easier to apply (which, coincidently, again gives the same estimate).

**Example:**

By using the Maclaurin series for \( e^x \), approximate the number \( e \) with an error less than 0.001.

**Method**

Take \( f(x) = e^x \) and \( a = 0 \), then calculate the Maclaurin polynomials, \( p_n(x) \), for \( f \), evaluate these at 1 (i.e. calculate \( p_n(1) \)) to obtain approximations for \( f(1) = e^1 = e \) and finally estimate the error using the Lagrange form of the remainder.

\( f^{(n)}(x) = e^x \) for all \( n \) and for all \( x \) and so \( f^{(n)}(0) = 1 \) for all \( n \). Then

\[
p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}.
\]

The approximations are

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_n(1) )</td>
<td>1</td>
<td>2</td>
<td>2.5</td>
<td>2.667</td>
<td>2.708</td>
<td>2.717</td>
<td>2.178</td>
</tr>
</tbody>
</table>

The error in using the approximation \( p_n(1) \) is given by \( |R_n(1)| \) and by the Lagrange form of the remainder

\[
R_n(1) = \frac{f^{(n+1)}(c)}{n!} 1^{n+1} = \frac{e^c}{(n+1)!}
\]

for some value of \( c \) between 0 and 1.

Since \( 0 < c < 1 \) and \( f(x) = e^x \) is an increasing function, \( 1 = e^0 < e^c < e^1 = e < 3 \). So that

\[
|R_n(1)| \leq \frac{3}{(n+1)!}.
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3}{(n+1)!} )</td>
<td>3</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{24} )</td>
<td>( \frac{3}{120} )</td>
<td>( \frac{3}{720} )</td>
<td>( \frac{3}{5040} )</td>
</tr>
</tbody>
</table>

From the table it can be seen that for the required accuracy we need to take \( n = 6 \). This gives the approximation for \( e \) of \( \frac{1957}{720} \).
8.4.3 Convergence of Taylor Series using Remainders

An important step in the estimate for \( R_6(1) \), for \( \sin x \), in the last section was the inequality \(|\cos c| \leq 1\), which holds no matter what the value of \( c \). Since the \( n \)th derivative of \( \sin x \) is \( \pm \sin x \) or \( \pm \cos x \) the same inequality will apply to any derivative of \( \sin x \). That is

\[
|f^{(n+1)}(c)| \leq 1
\]

for any value of \( n \) and any value of \( c \).

Using this inequality in the Lagrange form of the remainder for the Maclaurin series for \( \sin x \) shows that

\[
|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.
\]

No matter what value is taken for \( x \), the right hand side of this inequality converges to 0 as \( n \) tends to \( \infty \). That is

\[
\lim_{n \to \infty} R_n(x) = 0 \text{ for all } x, \quad \text{or equivalently, } \quad \sin x = f(x) = \lim_{n \to \infty} p_n(x) \text{ for all } x.
\]

**Definition:**

Given a function \( f \), we say that the Taylor Series for \( f \) with centre \( a \) converges to \( f \) at the point \( x \) if

\[
\lim_{n \to \infty} R_n(x) = 0,
\]

and in this case we write

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.
\]

Note that an equivalent condition is \( f(x) = \lim_{n \to \infty} p_n(x) \).

**Example:**

From the above calculation for the remainder, the Maclaurin series for \( \sin x \) converges to \( \sin x \) for all \( x \) and we can write

\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.
\]

In the table in Section 8.2.3 the last column gives the values of \( x \) for which each Maclaurin series converges to the given function.

8.4.4 \( O(h) \) ERROR

Numerical analysis introduces errors due to rounding off, limitations of the computer and/or approximations done for analytic purposes. Numerical errors will usually depend on some parameter, say, \( h \) where \( h \) is small.
Let \( f \) be a function of \( h \). Then \( f(h) \) converges to \( L \) with a rate of convergence of \( O(h) \) when

\[
\text{error} = f(h) - L = O(h) \quad \text{as} \quad h \to 0
\]

means that the error behaves like \( Mh \) for small \( h \), where \( M \) is a constant and finite.

That is,

\[
\lim_{h \to 0} \frac{\text{error}}{h} = M.
\]

Note: In the numerical analysis sense, usually, \( h \) is step or interval length between two consecutive spatial values.

If

\[
\text{error} = O(h^n) \quad \text{as} \quad h \to 0
\]

we mean that the error behaves like \( Mh^n \) for small \( h \). That is, \( f(h) \) converges to \( L \) with a rate of convergence of \( O(h^n) \)

That is,

\[
\lim_{h \to 0} \frac{\text{error}}{h^n} = M.
\]

Alternatively, we say that

Note

If \( n = 1 \), we say the error behaves linearly.

If \( n = 2 \), we say the error behaves quadratically.

Example

Find the rate of convergence for \( \sin h \), for \( h \to 0 \).

Method

We know that

\[
\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} + \ldots, \quad \text{and} \quad \lim_{h \to 0} \sin(h) = L = 0.
\]

Therefore,

\[
\lim_{h \to 0} \frac{\text{error}}{h^n} = \lim_{h \to 0} \frac{|\text{True value - approx. value}|}{h^n} = \lim_{h \to 0} \frac{0 - \sin(h)}{h^n} = \lim_{h \to 0} \left( \frac{0 - \left( h - \frac{h^3}{3!} + \frac{h^5}{5!} + \ldots \right)}{h^n} \right) = \lim_{h \to 0} \left( \frac{h - \frac{h^3}{3!} + \frac{h^5}{5!} + \ldots}{h^n} \right).
\]

For convergence, we require that \( n = 1 \). Hence,

\[
\lim_{h \to 0} \frac{0 - \sin(h)}{h} = \lim_{h \to 0} \left( 1 - \frac{h^2}{3!} + \frac{h^4}{5!} + \ldots \right),
\]

\[
= 1 \quad (\text{i.e. } M).
\]
Thus, the error behave like $h$ for small values. That is,

$$\text{error} = O(h) \quad \text{or} \quad \sin(h) = O(h).$$

**Exercise 8A**

1. State the number of significant figures in each of the following:
   
   (a) 465 \hspace{1cm} (b) 0.0003 \hspace{1cm} (c) 63700
   
   (d) 405.03 \hspace{1cm} (e) 37.0 \hspace{1cm} (f) 0.032015

2. Round off errors can become particularly disadvantageous in the expression $a - b$, when $a$ and $b$ are close together.
   
   To illustrate this calculate using the numbers given and then rounding off to 4, 2 significant digits.

3. Generate the Taylor series expansion for each of the following functions about the given point.
   
   (a) $e^x$ about $a = 1$
   
   (b) $\sin x$ about $a = \frac{\pi}{6}$
   
   (c) $\cos x$ about $a = \frac{\pi}{6}$
   
   (d) $\ln x$ about $a = 1$

4. Find the Maclaurin polynomials $p_1$, $p_3$ and $p_5$ for $f(x) = \sin x$. Using a calculator, if needed, evaluate each of these polynomials at the points 0, 0.25, 0.5, 0.75 and 1. Compare the values obtained with the corresponding values for $f(x)$.

5. Find the Taylor polynomials $p_1$ and $p_4$ for $\ln x$ with centre 1 and evaluate them at 1, 1.25, 1.5, 1.75 and 2. Compare with the values of $\ln x$.

6. Find Lagrange's form of the Remainder, $R_n(x)$, for the following functions $f$, with the given degree $n$ and the given centre $a$. Further, give an upper bound on the maximum error that will result when using the Taylor polynomial to approximate the function in the given interval.
   
   (a) $f(x) = \ln(x-1)$, $n = 3$, $a = 2$, $[1.5, 2]$.
   
   (b) $f(x) = e^{x/2}$, $n = 4$, $a = 0$, $[-1, 0]$.
   
   (c) $f(x) = \sqrt{x}$, $n = 2$, $a = 1$, $[\frac{4}{3}, 1]$.

7. Prove that the Maclaurin series for the function $f(x) = e^x$, which converges for all $x$, converges to the function.

8. For what values of $x$ does the approximation $\ln(1 + x) \approx x$ give two decimal place accuracy? You should consider the cases $x \geq 0$ and $x < 0$ separately.

9. Use the series for $\ln \frac{1 + x}{1 - x}$ to find $\ln 2$ to three decimal places.

10. Find the rate of convergence for the functions as $h \to 0$.
   
   (a) $\cos h$ \hspace{1cm} (b) $\frac{\sin^2 h}{h^2}$ \hspace{1cm} (c) $\frac{1 - \cos h}{h^2}$
   
   (d) $e^{h^2} - 1$ \hspace{1cm} (e) $\tan h - h$.

**8.5 NUMERICAL DIFFERENTIATION**

Consider the Taylor series expansion of $f(x)$ about the point $x = x_i$, then

$$f(x) = \sum_{n=0}^{\infty} (x - x_i)^n f^{(n)}(x_i) = f(x_i) + (x - x_i)f'(x_i) + \frac{(x - x_i)^2}{2} f''(x_i) + O((x - x_i)^3).$$
Notation

We shall introduce the notation that

\[ f_i = f(x_i) \quad f'_i = f'(x_i) \quad \text{and} \quad f''_i = f''(x_i). \]

Then,

\[ f_{i-1} = f(x_{i-1}) \quad \text{and} \quad f_{i+1} = f(x_{i+1}) \]
\[ f'_{i-1} = f'(x_{i-1}) \quad \text{and} \quad f'_{i+1} = f'(x_{i+1}) \]
\[ f''_{i-1} = f''(x_{i-1}) \quad \text{and} \quad f''_{i+1} = f''(x_{i+1}), \quad \text{etc.} \]

That is, the subscript represents the value of the function and the respective \( x \) value.

Usually we are only given a data set for a given function. With this in mind it is possible to obtain a linear and quadratic approximation of the original function from the given data set by determining appropriate approximations to the first and second derivatives. This is shown in the next sections.

8.5.1 First Derivative Approximation for a Function

We can obtain an approximation to the first derivative term using (1) by the following procedure.

Using (1), substitute \( x = x_{i+1} \) and \( x = x_{i-1} \), respectively, to obtain the following equations:

\[ f_{i+1} = f_i + (x_{i+1} - x_i)f'_i + \frac{(x_{i+1} - x_i)^2}{2}f''_i + O((x_{i+1} - x_i)^3). \] \( (2) \)

and

\[ f_{i-1} = f_i + (x_{i-1} - x_i)f'_i + \frac{(x_{i-1} - x_i)^2}{2}f''_i + O((x_{i-1} - x_i)^3). \] \( (3) \)

Remember that \( O((x_{i-1} - x_i)^3) \) is an expression of the rate of convergence.

Using the fact that \( h = x_{i+1} - x_i \) and \( -h = x_{i-1} - x_i \) then (2) and (3) become

\[ f_{i+1} = f_i + hf'_i + \frac{h^2}{2}f''_i + O(h^3). \] \( (4) \)

and

\[ f_{i-1} = f_i - hf'_i + \frac{h^2}{2}f''_i - O(h^3). \] \( (5) \)

Subtracting (5) from (4) gives

\[ f_{i+1} - f_{i-1} = 2hf'_i + O(h^3) \]
then

\[ f'_i = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2). \]

Noting that \( \frac{O(h^3)}{h} = O(h^2) \). Hence, an approximation to the first derivative of \( f(x) \) at \( x = x_i \) is

\[ f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h}. \] \( (6) \)

with error \( = O(h^2) \). The right-hand side of (6) is called the difference quotient.
8.5.2 Second Derivative Approximation of a Function

We can obtain an approximation to the second derivative term using (1) by adopting the following procedure. This procedure is similar to that used for the finding an approximation to the first derivative term.

To obtain an approximation to the second derivative term we will take a further term in the Taylor series expansion of \( f(x) \) about the point \( x_i \), then

\[
f(x) = f(x_i) + (x - x_i)f'(x_i) + \frac{(x - x_i)^2}{2} f''(x_i) + \frac{(x - x_i)^3}{6} f'''(x_i) + O ((x - x_i)^4).
\]

(7)

Note: We take more terms in the Taylor series expansion as we find that the third term cancels and therefore we have a bonus of an extra term. This gives an improved accuracy for the second derivative approximation.

Using (7), substitute \( x = x_{i+1} \) and \( x = x_{i-1} \), respectively, to obtain

\[
f_{i+1} = f_i + hf_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{6} f_i''' + O (h^4)
\]

(8)

and

\[
f_{i-1} = f_i - hf_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i''' + O (h^4)
\]

(9)

Adding (8) and (9) gives that

\[
f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + O (h^2)
\]

where \( \frac{O (h^4)}{h^2} = O (h^2) \).

Hence, an approximation to the second derivative of \( f(x) \) at \( x = x_i \) is

\[
f''(x_i) \approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}.
\]

(10)

with error = \( O (h^2) \). The right-hand side of (10) is called the difference quotient.

8.5.3 Approximation of a Function without Derivative Terms

Suppose we wish to use the first three terms of a Taylor series expression to find an approximation of \( f \) at \( x \) near \( x = x_i \), given the values of \( f \) at \( x = x_{i-1} \), \( x = x_i \) and \( x = x_{i+1} \) then

\[
f(x) \approx f(x_i) + (x - x_i)^2f'(x_i) + (x - x_i)f''(x_i)
\]

\[
= f(x_i) + (x - x_i) \left( \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \right) + \frac{(x - x_i)^2}{2} \left( \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \right).
\]

is of \( O (h^3) \).
Example

(a) Let \( f(x) = \tan^{-1}(x^2) \). Use an approximation method to evaluate \( \frac{df}{dx} \) at \( x = 1 \) correct to \( O(h^2) \) when the step length is \( h = 0.1 \). Hence, find a linear approximation to \( f(x) \) near 1.

Method

Recall that for \( O(h^2) \),

\[
f'_i \approx \frac{f_{i+1} - f_{i-1}}{2h}
\]

Let \( x_i = 1 \), then \( x_{i-1} = x_i - h = 0.9 \) and \( x_{i+1} = x_i + h = 10.1 \). Therefore, we require that we know

\[
\begin{align*}
f_{i-1} &= f(0.9) = \tan^{-1}(0.9)^2 = 0.6808 \\
f_i &= f(1) = \tan^{-1}(1)^2 = \frac{\pi}{4} = 0.7854, \\
f_{i+1} &= f(10.1) = \tan^{-1}(10.1)^2 = 1.5610.
\end{align*}
\]

Hence,

\[
\frac{df}{dx} \approx \frac{f(10.1) - f(0.9)}{2 \times 0.1} = \frac{1.5610 - 0.6808}{0.2} = 4.4010.
\]

Therefore, a linear approximation to \( f(x) \) at \( x_i = 1 \) is

\[
f(x) \approx f_i + (x - x_i)f'_i = 0.7854 + 4.4010(x - 1)
\]

\[= 4.401x - 3.6156.
\]

(b) Given \( f(x) = \ln x \), \( \ln(9.0) = 2.1972 \), \( \ln(9.5) = 2.2513 \) and \( \ln(10.0) = 2.3026 \).

Use a quadratic approximation to find an approximation to \( \ln(9.2) \).

Method

Given the values of \( x \) are at 9.0, 9.5 and 10.0, respectively, then it easy to assume that we let \( x_{i-1} = 9.0 \), \( x_i = 9.5 \) and \( x_{i+1} = 10.0 \). Therefore, \( h = 0.5 \). Hence,

\[
f'_i \approx \frac{f_{i+1} - f_{i-1}}{2h} = \frac{\ln(10.0) - \ln(9)}{2 \times 0.5} = 0.1054.
\]

and

\[
f''_i \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} = \frac{\ln(10.0) - 2 \ln(9.5) + \ln(9.0)}{(0.5)^2} = -0.0111.
\]

Therefore, the quadratic approximation is obtained from

\[
f(x) \approx f(x_i) + (x - x_i)f'(x_i) + \frac{(x - x_i)^2}{2}f''(x_i).
\]
Substituting $x_i = 9.5$ gives

$$f(x) \approx f(9.5) + (x - 9.5)f'(9.5) + \frac{(x - 9.5)^2}{2}f''(9.5).$$

Let $x = 9.2$ then

$$f(9.2) = \ln(9.2) \approx f(9.5) + (9.2 - 9.5)f'(9.5) + \frac{(9.2 - 9.5)^2}{2}f''(9.5)$$
$$= 2.2513 - 0.3 \times (0.1054) + 0.045 \times (-0.0111)$$
$$= 2.2202$$

Error in this case is $O((0.5)^3) = O(0.125)$. Note the calculator value is 2.2192.

### 8.5.4 Differences Operators

The term $f(x_{i+1}) - f(x_i)$ is called the **forward difference**. The operator that is connected with this forward difference is $\Delta$ and the **first forward difference** is obtained from

$$\Delta f_i = f_{i+1} - f_i = f(x_{i+1}) - f(x_i).$$

The second forward difference is written as

$$\Delta^2 f_i = \Delta(\Delta(f_i)) = \Delta(f_{i+1} - f_i) = \Delta(f_{i+1}) - \Delta(f_i) = f(x_{i+2}) - f(x_{i+1}) - (f(x_{i+1}) - f(x_i)) = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i).$$

The term $f(x_i) - f(x_{i-1})$ is called the **backward difference**. The operator that is connected with this backward difference is $\nabla$. Hence,

$$\nabla f_i = f_{i-1} - f_i = f(x_{i-1}) - f(x_i).$$

The term $f(x_{i+1}) - f(x_{i-1})$ and $f(x_{i+1}) - 2f(x_i) + f(x_{i-1})$ are called **central differences**. The central difference operator is symbolised by $\delta$.

### 8.6 NUMERICAL INTEGRATION

#### 8.6.1 Introduction

This section comprises some numerical evaluation techniques for calculating approximations to the definite integral

$$I = \int_a^b f(x) \, dx.$$
In most of what follows we will restrict ourselves to a discussion of integration on the interval \([0, 1]\). This makes some of the arithmetic easier. We note that the change of range can be achieved by a simple change of variable in the integral. That is,

\[
x = a + (b - a)y
\]
giving

\[
dx = (b - a)dy
\]
and

\[
x = a \implies y = 0 \quad x = b \implies y = 1.
\]

Therefore,

\[
I = \int_0^1 f[a + (b - a)y](b - a) \, dy
\]

\[
= (b - a) \int_0^1 f[a + (b - a)y] \, dy.
\]

We will consider a numerical integration, or quadrature rule which gives us an approximation of the form

\[
\int_0^1 f(x) \, dx \approx \sum_{n=1}^N w_i f(x_i)
\]

where the constants \( \{w_i\} \) are usually called weights or coefficients, and the values \( \{x_i\} \) at which \( f(x) \) is evaluated, are called ordinates or sample points, of the rule.

Such a rule cannot be exact for every type of function \( f(x) \) which we might wish to integrate, so in order to determine the values of \( \{x_i\} \) and \( \{w_i\} \), we require that the formula be exact for some simple set of functions. We choose \( 1, x, x^2, \ldots \) because these functions are easy to use in calculations, and also, most functions can be approximated well by a low degree polynomial over a small interval in \( x \).

### 8.6.2 The Midpoint Rule

The simplest possible rule would be a one point rule:

\[
\int_0^1 f(x) \, dx \approx w_1 f(x_1).
\]

This rule has two unknowns, \( w_1 \) and \( x_1 \), which we can choose so that the rule is as good as possible. We should be able to integrate the first two powers of \( x \) exactly.

**Consider \( f(x) = x^0 = 1 \):**

Ordinary integration gives

\[
\int_0^1 f(x) \, dx = \int_0^1 1 \, dx = \left[ x \right]_0^1 = 1 - 0 = 1
\]

Using our rule,

\[
\int_0^1 f(x) \, dx = w_1 f(x_1)
\]

we obtain

\[
\int_0^1 x^0 \, dx = w_1 (x_1)^0 = w_1.
\]
Hence, using the function $f(x) = 1$, we equate these results to obtain the equation

$$w_1 = 1 \quad \ldots (1)$$

Consider $f(x) = x$:

Ordinary integration gives

$$\int_0^1 f(x) \, dx = \int_0^1 x \, dx = \left[ \frac{x^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$  

Using our rule, we obtain

$$\int_0^1 f(x) \, dx = w_1 f(x_1) = w_1 x_1 .$$

Hence, using the function $f(x) = x$, we equate these results to obtain the equation

$$w_1 x_1 = \frac{1}{2} \quad \ldots (2)$$

Combining the two equations and solving, we have

$$w_1 = 1 \quad w_1 x_1 = \frac{1}{2} \implies x_1 = \frac{1}{2}$$

so the rule is found to be

**Midpoint Rule on $[0,1]$**:  

$$\int_0^1 f(x) \, dx \approx f \left( \frac{1}{2} \right).$$

Translated back onto the interval $[a,b]$, we have

$$\int_a^b f(x) \, dx = (b - a) \int_0^1 f[a + (b - a)y] \, dy = (b - a) f \left( \frac{a + \frac{1}{2} (b - a)}{2} \right) \approx (b - a) f \left( \frac{a + b}{2} \right).$$

**Midpoint Rule on $[a,b]$**:  

$$\int_a^b f(x) \, dx \approx (b - a) f \left( \frac{a + b}{2} \right).$$

Geometrically, we are approximating the area under the curve by the rectangle shown.

Since the rule can integrate both 1 and $x$ correctly it can integrate exactly any linear function $f(x) = a_0 + a_1 x$.

To see what happens if we try to integrate a quadratic function, we can integrate $x^2$. 

Using ordinary integration: From the rule:

\[ \int_0^1 x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}. \]

\[ \int_0^1 x^2 \, dx = f \left( \frac{1}{2} \right) = \frac{1}{4}. \]

Thus the midpoint rule cannot integrate a quadratic function exactly.

### 8.6.3 The Elementary Trapezoidal Rule

To derive the elementary trapezoidal rule, we specify that the function is to be evaluated at the end points of the interval, giving the form

\[ \int_0^1 f(x) \, dx \approx w_1 f(0) + w_2 f(1). \]

Since there are 2 unknown weights, \( w_1 \) and \( w_2 \), we should be able to choose them so that the functions 1 and \( x \) can be integrated exactly.

**\( f(x) = 1: \)** \( \int_0^1 1 \, dx = 1 = w_1 \times 1 + w_2 \times 1 \)

**\( f(x) = x: \)** \( \int_0^1 x \, dx = \frac{1}{2} = w_1 \times 0 + w_2 \times 1. \)

Solving these equations, we have \( w_1 = w_2 = \frac{1}{2} \) so that the rule becomes

\[ \int_0^1 f(x) \, dx \approx \frac{1}{2} [f(0) + f(1)]. \]

The trapezoidal rule cannot exactly integrate \( x^2 \). It gives

\[ \int_0^1 x^2 \, dx = \frac{1}{2} [0 + 1] = \frac{1}{2} \quad \text{while we should have} \quad \int_0^1 x^2 \, dx = \frac{1}{3} \left( \neq \frac{1}{2} \right). \]

In this respect it is no better than the midpoint rule which needs one less function evaluation.

When the trapezoidal rule is translated onto the interval \((a, b)\) we have the familiar

\[ \int_a^b f(x) \, dx = (b - a) \int_0^1 f[a + (b - a)y] \, dy \]

\[ \approx (b - a) \frac{1}{2} [f(a) + f(b)]. \]

Thus the elementary trapezoidal rule is given by

**Elementary Trapezoidal Rule:**

\[ \int_a^b f(x) \, dx \approx (b - a) \frac{1}{2} [f(a) + f(b)]. \]
Geometrically, the area under the curve is approximated by the area of the trapezium.

8.6.4 The Elementary Simpson's Rule

To obtain Simpson's rule, we choose the rule to evaluate the function at \( x_1 = 0 \), \( x_2 = \frac{1}{2} \) and \( x_3 = 1 \), and we find the corresponding weights \( \{w_i\} \). Thus

\[
\int_0^1 f(x) \, dx = w_1 f(0) + w_2 f\left(\frac{1}{2}\right) + w_3 f(1).
\]

Since there are three unknown weights, we should be able to choose them so that we can integrate \( 1 \), \( x \) and \( x^2 \) successfully.

\[
\int_0^1 1 \, dx = 1 = w_1 + w_2 + w_3
\]
\[
\int_0^1 x \, dx = \frac{1}{2} = w_1 \times 0 + w_2 \times \frac{1}{2} + w_3 \times 1
\]
\[
\int_0^1 x^2 \, dx = \frac{1}{3} = w_1 \times 0 + w_2 \times \frac{1}{4} + w_3 \times 1.
\]

Solving these equations for \( w_1 \), \( w_2 \), \( w_3 \) we obtain

\[
w_1 = w_3 = \frac{1}{6} \quad \text{and} \quad w_2 = \frac{4}{6}
\]

so that the rule is

\[
\int_0^1 f(x) \, dx = \frac{1}{6} \left[ f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right].
\]

Surprisingly, this rule also integrates \( x^3 \) exactly:

\[
\int_0^1 x^3 \, dx = \frac{1}{4} \quad \text{by ordinary integration, and}
\]
\[
\int_0^1 x^3 \, dx = \frac{1}{6} \left[ 0 + \frac{4}{8} + 1 \right] = \frac{1}{4} \quad \text{by the rule.}
\]

Translated onto the interval \((a,b)\) the rule becomes

**Elementary Simpson's Rule:**

\[
\int_a^b f(x) \, dx \approx \frac{1}{3} \left( \frac{b-a}{2} \right) \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].
\]

It should be noted that when using the elementary Simpson's rule there is one interval but two subintervals. Thus, the length of each subinterval is \( h = \frac{(b-1)}{2} \). \( h \) is also known as the step length.
Extensions of the Simple Rules.

We have seen how various familiar rules can be derived using the same fundamental approach, namely, the demand to integrate \( 1, x, x^2, \ldots \) as far as possible. It is unlikely that any of the rules we have discussed so far would be adequate as they stand to integrate an arbitrary function on an interval \((a, b)\).

To remedy this situation we can do one of two things:

(a) derive a more complicated rule which can integrate polynomials of higher degree;

(b) subdivide the range of the integral, and apply a simpler rule to each of the pieces.

Each solution has merits for particular situations. Higher order rules are often used in finite element calculations in engineering, when the functions to be integrated are in fact polynomials of high degree. Subdivision of the range is usually employed for integrating arbitrary functions to a specified accuracy.

However, the question of accuracy is a difficult one. Theoretical estimates of accuracy can be obtained using Taylor’s series expansions. A practical approach is to compute an approximation to an integral, subdivide the range, and recompute the result, and repeat until the answers agree. It turns out that the humble trapezoidal rule, in tandem with a procedure called Richardson extrapolation, is a powerful integration technique.

Extensions to the simple rules are discussed in the following sections.

8.6.5 General Compound Trapezoidal and Simpson’s Rules

In the next section we will see how we can proceed with repeated halving of intervals, following an original application of the trapezoidal rule on the interval \([a, b]\), to obtain successively more accurate approximations for the value of an integral.

Subdivisions based on halving previous intervals are usually the easiest to apply, and often lead to the nice results which we will see in the following sections. However, sometimes we wish to apply an extended trapezoidal rule or Simpson’s rule with a general method of subdivision of a given interval.

Let the interval \([a, b]\) be divided into \(n\) equal subdivisions of length \(h\). Thus

\[ nh = b - a. \]

Then if we apply the trapezoidal rule to each of the intervals

\[ [a + (i - 1)h, a + ih] \quad i = 1, \ldots, n, \]

we can approximate \(\int_a^b f(x) \, dx\) by the summation
In this notation the elementary Simpson’s rule would be regarded as using two subintervals of length \( h \) each, where the pattern of coefficients is \( 1 4 2 4 2 \ldots \). Hence, the length (or step length) of each subinterval is \( h = (b - 1)/2n \). We can sum separate simple Simpson’s rules to obtain

\[
\int_a^b f(x) \, dx = \sum_{i=1}^n \int_{a+i/h}^{a+ih} f(x) \, dx
\]

\[
= \sum_{i=1}^n \frac{1}{2} h \{ f(a + [i-1]h) + f(a + ih) \}
\]

\[
= \frac{h}{2} \{ f(a) + 2f(a + h) + 2f(a + 2h) + \ldots + 2f(a + [n-1]h) + f(a + nh) \}
\]

to give

**Compound Trapezoidal Rule:**

\[
\int_a^b f(x) \, dx \approx \frac{h}{2} \{ f(a) + 2f(a + h) + 2f(a + 2h) + \ldots + 2f(a + [n-1]h) + f(a + nh) \}
\]

Similarly, if we again divide the interval \((a, b)\) into \( n \) equal intervals, then there will be \( 2n \) subintervals (an even number). Hence, the length (or step length) of each subinterval is \( h = (b - 1)/2n \). We can sum separate simple Simpson’s rules to obtain

**Compound Simpson’s Rule:**

\[
\int_a^b f(x) \, dx \approx \frac{h}{3} \{ f(a) + 4f(a + h) + 2f(a + 2h) + \ldots + 4f(a + [n-1]h) + f(b) \}
\]

where the pattern of coefficients is \( 1 4 2 4 2 \ldots 2 4 2 4 1 \).

In this notation the elementary Simpson’s rule would be regarded as using two subintervals of length \( h = (b - a)/2 \).

**Exercise 8B**

1. Devise a one point quadrature scheme

\[
\int_0^1 f(x) \, dx = w_1 f(x_1) .
\]

(i) Evaluate \( w_1 \) and \( x_1 \) so that the functions \( f(x) = 1 \) and \( f(x) = x \) are correctly integrated.

Can \( x^2 \) be correctly integrated?

(ii) Evaluate \( w_1 \) and \( x_1 \) so that the even functions \( f(x) = 1 \) and \( f(x) = x^2 \) are correctly integrated.

Can \( x^4 \) be correctly integrated? What general polynomial can be correctly integrated?

2. Trapezoidal integration uses

\[
\int_0^1 f(x) \, dx = w_1 f(0) + w_2 f(1) .
\]

Show that if the functions 1 and \( x \) are to be correctly integrated, then \( w_1 = w_2 = \frac{1}{2} \).

Show how this is related to the result

\[
\int_a^b f(x) \, dx = \frac{b - a}{2} [ f(a) + f(b) ] .
\]

3. \[
\int_0^1 f(x) \, dx = w_1 f(0) + w_2 f(0.5) + w_3 f(1)
\]
defines a quadrature rule.

Show that if the functions 1, \( x, x^2 \) are to be integrated correctly then \( w_1, w_2, w_3 \) are \( \frac{1}{6}, \frac{4}{6}, \frac{1}{6} \) respectively.

Show that this quadrature scheme also integrates \( x^3 \) exactly, a bonus for Simpson’s rule.

Show how your result is related to the simple Simpson’s rule

\[
\int_a^b f(x) \, dx = \frac{h}{3} \{ f(a) + 4f(a + h) + f(b) \},
\]

where \( 2h = b - a \).

Derive the compound Simpson’s rule

\[
\int_a^b f(x) \, dx \approx \frac{h}{3} \{ f(a) + 4f(a + h) + 2f(a + 2h) + \ldots + 4f(a + [n-1]h) + f(b) \}
\]

where \( h = (b - a)/2n \).

*continued next page...*
4. Using Simpson’s rule with 2 subintervals, and then with 4 subintervals, evaluate \[
\int_1^2 \frac{e^x - 1}{x} \, dx.
\]

5. Use Simpson’s rule with 2, 4, and 10 subintervals to evaluate \[
\int_1^2 \frac{1}{x} \, dx.
\]

6. Evaluate \[
\int_{\frac{x}{2}}^{x} f(x) \, dx,
\]
    using both the compound trapezoidal rule and the compound Simpson’s rule on the following tabulated data.

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>11</td>
<td>31</td>
<td>69</td>
<td>131</td>
<td>223</td>
</tr>
</tbody>
</table>

7. The values of a polynomial \( f \) are given at the points \( x = 0, 1, 2, 3, 4, 5, 6 \), as follows:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-5</td>
<td>-2</td>
<td>7</td>
<td>67</td>
<td>130</td>
<td>223</td>
<td></td>
</tr>
</tbody>
</table>

(a) Using these values, apply trapezoidal rules in order to verify that the values 256, 184, 166, are approximations to the integral \[
\int_1^5 f(x) \, dx,
\]
    using one step of length 4, using two steps of length 2, and using four steps of length 1.

(b) What is value of \( \int_1^5 f(x) \, dx \) given by Simpson’s Rule, using two steps of length 2, and using four steps of length 1.

(c) Use Simpson’s rule with steps of length 1 to evaluate an approximation to \( \int_0^6 f(x) \, dx \).

8.7 FINDING ROOTS OF AN EQUATION

There are various methods for finding the roots of an equation both analytically and numerically. Each method has its advantages and disadvantages. The method of finding a root of an equation is highly dependent on the function and the required order of accuracy. Two well known methods, namely, Picard and Newton’s method will be outlined here. For detailed information concerning these methods and other methods, the reader should refer to the many numerical methods books that are available.

Consider the equation \( f(x) = 0 \).  

(*)

where \( f(x) \) is a continuous function.

For instance, we are trying to find where \( y = f(x) \) cuts the \( x \) axis. That is the root of \( y = f(x) \).

\[
\begin{align*}
\text{In the figure above the root of } y = f(x) = 0 \text{ is at } x = c. 
\end{align*}
\]
The following methods can be used to gain a good approximation to the roots of equation (*).

### 8.7.1 Direct (Picard’s) Iteration Scheme

Let an approximation to the real root of equation (*) be \( x_0 \) and the real root be \( X \).

The direct iteration method rearranges (*) into the form

\[
x = g(x).
\]

Geometrically, (1) represents the intersection of the line \( y = x \) and \( y = g(x) \), where \( g(x) \) is a differentiable function near the point of intersection. From (1), an iteration scheme can be devised and is written in the form:

\[
x_{n+1} = g(x_n) \quad n = 1, 2, 3, \ldots.
\]

There are many choices for the function \( g(x) \). However, for the iteration scheme to be absolutely convergent

\[|g'(x)| < 1.\]

Therefore, if the scheme is convergent then the figure below explains what happens as \( n \) increases.

This condition can be proved by the use of a Taylor Series expansion. The above figure is called a cobweb diagram.

The figure below explains graphically what happens when there is no convergence of the scheme \( x = g(x) \).
Note:
An approximation to the initial value $x_0$ can be obtained by approximating the intersection of the line $y = x$ and the curve $y = g(x)$. This can be done using Matlab.

The direct iteration (or Picard’s) method can be shown by the following example.

Example

Find a root of the equation $f(x) = 8x^3 + 12x^2 + 130x + 63 = 0$ near $x_0 = 0$.

Method

Set up an iteration scheme in the form $x_{n+1} = g(x_n)$ so that the convergence criteria $|g'(x_0)| < 1$ is satisfied. Therefore, $f(x)$ is rearranged into the form $x = g(x)$ where $g(x) = \frac{63}{8x^2 + 12x + 130}$ has been chosen so that $|g'(0)| < 1$.

Given $x_0 = 0$ and let $n = 0$, (2) becomes

$$x_1 = g(x_0) = g(0) = \frac{63}{130} = 0.4846$$
corrected to 4 decimal places.

Given that $x_1 = 0.4846$ and $n = 1$, (2) becomes

$$x_2 = g(x_1) = g(0.4846) = 0.45754$$

Repeating the process by letting $n = 3, \ldots$, we have that

$$x_3 = g(x_2) = g(0.45754) = 0.45930$$
$$x_4 = g(x_3) = g(0.45930) = 0.45919$$
$$x_5 = g(x_4) = g(0.45919) = 0.45919$$
corrected to 4 decimal places.

It can be seen from the last two iteration giving $x_4$ and $x_5$, respectively that the $x$ values are the same. Therefore, the iteration scheme has reached convergence and as a result we have found the root of $f(x) = 0$. This root being $x = 0.45919$ corrected to 4 decimal places. Of course, if further accuracy is required then further iterations will be necessary.
8.7.2 Newton’s Method

Newton’s Method is used to find the roots of (*) provided that the function \( f(x) \) is differentiable at least once. The iteration scheme is

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots
\]

Geometrically, the iteration scheme in (3) finds successive approximations to the slope of a line that is tangent to the curve \( y = f(x) \). If the iteration scheme converges then the slope of the line will be the tangent of the curve at a root of (*).

The initial guess for the schemes is at \( x = x_0 \). Graphically,

Newton’s method converges rapidly to the root of the equation \( f(x) = 0 \) provided the initial approximation is close to this root. Otherwise, the scheme may or may not converge. A disadvantage of Newton’s method is that it requires the derivative of the given function.

8.8 FIRST ORDER INITIAL VALUE PROBLEM

Consider the initial-value problem of the form

\[
\frac{dy}{dx} = f(x, y)
\]

subject to

\[ y(x_0) = y_0. \]

Note

- The independent variable is \( x \) and the dependent variable is \( y \).

- The right-hand side of the differential equation can be a function of \( x \) and \( y \), or a function of \( x \) or \( y \) only.

One way to solve (1), numerically, is to use tangent line approximations for \( \frac{dy}{dx} \). However, to do this we are required to recall the term steplength or stepsize.
8.8.1 Derivative Approximation

Let \( h \) represent small change, or **stepsize** in the independent variable \( x \).

We approximate the value of the solution \( y \) in (1) at the sequence of \( x \)-values, \( x_1, x_2, x_3 \ldots \), such that

\[
x_n = x_{n-1} + h = x_0 + nh.
\]

Then the slope of the tangent line to the graph of the solution \( y \) at these values of \( x \) is found using the differential equation. That is, since

\[
\frac{dy}{dx} = f(x, y)
\]

then the slope of the tangent line at \( x = x_0 \) is \( f(x_0, y_0) \) where \( y_0 = y(x_0) \). This can be seen in the figure below.

![Solution Curve](image)

Hence, using the gradient point form, the equation of the tangent line to the graph of \( y \) at the point \((x_0, y_0)\) is

\[
\frac{y - y_0}{x - x_0} = f(x_0, y_0).
\]

Rewriting, we have that

\[
y = y_0 + (x - x_0)f(x_0, y_0). \tag{2}
\]

By setting \( x = x_1 \) and \( y = y_1 \) in (2), we can find the approximate value of \( y \) at \( x = x_1 \) (which we can call \( y_1 \)). Hence,

\[
y_1 = y_0 + (x_1 - x_0)f(x_0, y_0) = y_0 + hf(x_0, y_0) \tag{3}
\]

where \( h = x_1 - x_0 \) (the stepsize).

Thus we have obtained an approximate value of the solution \( y \) in (1) at \( x = x_1 \) (called \( y_1 \)). It can be seen from the figure below that the we are now obtaining an approximation to the true solution curve. Hence, this means that errors are introduced after the first approximation of the true solution. Consequently, there will be an increase in error as we increase the value of \( x \).
Since we have the point \((x_1, y_1)\) then we use this point to estimate the value of \(y\) when \(x = x_2\) (i.e. \(y = y_2\)). We use the same procedure as we did before. That is, using the gradient point form, the equation of the tangent line to the graph of \(y\) at the point \((x_1, y_1)\) is

\[
\frac{y - y_1}{x - x_1} = f(x_1, y_1).
\]

Rearranging this equation we have

\[
y = y_1 + (x - x_1)f(x_1, y_1).
\]

Substitute \(x = x_2\) and \(y = y_2\) then we have that

\[
y_2 = y_1 + (x_2 - x_1)f(x_1, y_1)
\]

\[
= y_1 + h f(x_1, y_1)
\]

where \(x_2 - x_1 = h = x_0 + 2h\). (the stepsize).

Hence, we have the point \((x_2, y_2) = (x_0 + 2h, y_2)\). The following figure depicts what Euler’s iteration scheme is generating. With each calculation there is a difference between the true solution value and the approximate solution. This difference is called the absolute error.
Continuing this procedure, we see that at \( x = x_n \), where \( x_n = x_0 + nh \) and

\[
y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).
\]  

(4)

Using (4), we obtain a sequence of points of the form

\[
\{(x_n, y_n)\} \quad n = 1, 2, 3, \ldots,
\]

where \( y_n \) is an approximate value of \( y(x_n) \). Several points of this type are shown in the figure below along with the actual value of \( y \).

This is the basis of the following two numerical methods or schemes.

### 8.9 Euler’s Method

The solution to the initial value problem

\[
y' = f(x, y) \quad y(x_0) = y_0
\]

is approximated at the sequence of points \((x_n, y_n)\) \( (n = 1, 2, 3, \ldots) \), where \( y_n \) is the approximate value of \( y(x_n) \) by computing

\[
y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}), \quad n = 1, 2, 3, \ldots,
\]  

(*)

where \( x_n = x_0 + nh \) and \( h \) is the selected stepsize.

**Example**

Use Euler’s method with step size \( h = 0.1 \) to approximate the solution of

\[
y' = xy, \quad y(0) = 1
\]

on \( 0 \leq x \leq 1 \). Also, determine the exact value and compare results.
Method

Note: In this example,

\[ f(x, y) = xy \]

the RHS of the differential equation

and the initial point is

\[ x_0 = 0 \quad y_0 = 1. \]

Using these facts and that \( h = 0.1 \), (*) becomes

\[ y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \]
\[ = y_{n-1} + (0.1) x_{n-1} y_{n-1} \quad n = 1, 2, 3 \ldots \]

To approximate \( y(x_n) \), let

\[ x_1 = x_0 + h \]
\[ = 0 + 0.1 \]
\[ = 0.1 \]

then

\[ y_1 = y_0 + (0.1) x_0 y_0 \]
\[ = 1 + (0.1) (0) (1) \]
\[ = 1.005 \]

Hence, we have the points

\[ (x_0, y_0) = (0, 1) \]

and the newly calculated point

\[ (x_1, y_1) = (0.1, 1) \]

where \( y_1 = 1 \) is an approximation to the true solution \( y(x_1) = y(0.1) \).

Now for

\[ x_2 = x_0 + 2h \]
\[ = 0.2, \]

we substitute \((x_1, y_1) = (0.1, 1)\) for \( n = 1 \) into (*) which gives

\[ y_2 = y_1 + 0.1 x_1 y_1 \]
\[ = 1 + 0.1 (0.1) (1) \]
\[ = 1.02. \]

Therefore, \( y_2 = 1.02 \) is an approximation to true solution \( y(x_2) = y(0.2) \).

Thus we have the point

\[ (x_2, y_2) = (x_0 + 2h, y_2) \]
\[ = (0.2, 1.02). \]

In the following table there is are results of the above sequence of approximations.
From this, we see that
\[ y_{10}(1) = 1.6479 \approx y(1). \]

Analytically, the solution to the initial value problem using separation of variables is
\[ y(x) = e^{x^2/2}. \]

So the exact value for \( y(1) \) is 1.6487.

From the table, it can be seen that a relative percentage error of 0.05% exists in calculating \( y(1) \) using Euler’s method. The table of values implies that the simple Euler’s scheme gives a reasonable accuracy to the solution. However, as the calculations move further away from the initial starting point the approximation to the true solution becomes worse.

Graphically,

As the above graph and table suggest there appears to be increasing errors in using Euler’s Method.

Can you give reasons for this inaccuracy?

How can we improve the accuracy of this method?

Let’s see what happens when we decrease the stepsize.
Let $h = 0.05$. We use

\[
y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) = y_{n-1} + hf(x_0 + nh, y_{n-1}) = y_{n-1} + 0.05 (x_{n-1}y_{n-1})
\]

to obtain the values given in the Table below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$y_n$</th>
<th>$y_n(x_n)$</th>
<th>Rel. % Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>1.0050</td>
<td>1.0050</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>1.0113</td>
<td>1.0113</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>1.0202</td>
<td>1.0202</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>1.0151</td>
<td>1.0151</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>1.0317</td>
<td>1.0317</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.3</td>
<td>1.0460</td>
<td>1.0460</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>1.0631</td>
<td>1.0632</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0.4</td>
<td>1.0832</td>
<td>1.0833</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0.45</td>
<td>1.1065</td>
<td>1.1066</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.5</td>
<td>1.1331</td>
<td>1.1331</td>
<td>0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>20</td>
<td>1.0</td>
<td>1.6485</td>
<td>1.6487</td>
<td>0.01</td>
</tr>
</tbody>
</table>

With stepsize $h = 0.05$, the approximate value of $y(1)$ is 1.5959. This is an error of 41% which is a ‘relatively’ smaller error than for the step size of $h = 0.1$.

Graphically,

![Graph](image)

So it ‘appears’ that decreasing the step size using Euler’s method improves the accuracy of the approximation. However, there is still a large discrepancy between Euler’s method and the true solution. As a result, better methods are required. Although, Euler’s method is useful when quick and easy calculations are required.
8.10 NUMERICAL ERRORS

There are several sources of error in Euler’s method and other numerical methods when determining a numerical approximation of $y(x_n)$. These errors are defined in the next sections.

8.10.1 Local Error

The linear approximation formula, namely,

$$y(x_{n+1}) = y_n + h f(x_n, y_n) = y_{n+1}. \quad (5)$$

Here $y_{n+1}$ is the amount the tangent line at $(x_n, y_n)$ departs from the solution curve through $(x_n, y_n)$ as shown in the diagram below.

The local error would be the total error in $y_{n+1}$ if the starting point $y_n$ in (5) were the exact value.

8.10.2 Cumulative Error

This is simply the addition of all the local errors together. Hence,

$$\text{Cumulative Error} = |y(x_n) - y_n|.$$

This error is sometimes called the **global truncation error** or global error. One way to decrease the size of the global error is to decrease the stepsize.

In general, it can be shown that if an approximate method is used to obtain a numerical solution for a differential equation with local truncation error of $O(h^{n+1})$ then the accumulative error is $O(h^n)$. 
8.10.3 Round-off Error

The computer itself will contribute round-off error at each stage because only a finitely number of significant digits can be used in each calculation. Therefore, the smaller the stepsize will introduce round-off error.

See Numerical Analysis books concerning error propagation.

8.10.4 The Error in Euler Method

The local trunction erro for the Euler’s scheme is of $O(h^2)$. Therefore, the global truncation error is $O(h)$.

‘Normally’ halving the stepsize interval cuts the maximum error in half.

8.11 IMPROVED EULER’S METHOD

This method is often called the Heun’s formula.

8.11.1 Method

Recall that Euler’s method uses the predicted slope

$$k = f(x,y)$$

of the graph of the solution at the left-hand endpoint of the interval

$$[x_n, x_n + h]$$

as if it were the actual slope of the solution over that entire interval. See graph below.
We can improve the accuracy of Euler’s method. This is shown below.

Consider the initial-value problem of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$  \hspace{1cm} (1)

Suppose we carry out \( n \) steps with stepsize \( h \). Hence, we have computed the approximation to \( y(x_n) \) of the solution at \( x_n = x_0 + nh \). To improve the estimate of \( y(x_{n+1}) \) we can do the following.

1. Use Euler’s method to obtain a first estimate. That is, find \( u_1 \) (say) where

\[
u_1 = y_0 + h f(x_0, y_0) = y_0 + h k_0.
\]

This is shown in the figure below.

Now \( u_1 \approx y(x_1) \).
2. Let

\[ k_1 = f(x_1, u_1) \]

be a second estimate of the slope of the solution curve \( y = y(x) \) at \( x = x_1 \).

3. Take the average of these two slopes to obtain a more accurate estimate of the slope of the curve over the interval \([x_0, x_1]\). This is shown graphically below.

4. Translate the averaged slope line with slope \( k_{\text{ave}} \) at the point \((x_1, u_1)\) to the point \((x_0, y_0)\). Hence, the intersection of the the line \( x = x_1 \) and this new slope line generates an improved value of \( y \), namely, \( y_1 \).

Generally,

1. Use Euler’s method to obtain a first estimate. That is, find \( u_{n+1} \) (say) where

\[ u_{n+1} = y_n + h f(x_n, y_n) = y_n + h k_1. \]

Now

\[ u_{n+1} \approx y(x_{n+1}). \]
2. Let

\[ k_2 = f(x_{n+1}, u_{n+1}) \]

be a second estimate of the slope of the solution curve \( y = y(x) \) at \( x = x_{n+1} \).

3. Take the average of these two slopes to obtain a more accurate estimate of the slope of the curve over the interval \( [x_n, x_{n+1}] \). This is shown graphically below.

Thus the algorithm for (1) is

\[
\begin{align*}
    k_1 &= f(x_n, y_n) \\
    u_{n+1} &= y_n + h \cdot k_1 \\
    k_2 &= f(x_{n+1}, u_{n+1}) \\
    y_{n+1} &= y_n + h \left( \frac{k_1 + k_2}{2} \right)
\end{align*}
\]

Note

- \( \frac{k_1 + k_2}{2} \) is the approximate average slope on the interval \( [x_n, x_{n+1}] \).
- The improved Euler method is one of a class of numerical techniques known as **predictor-corrector** methods. That is, we use the predictor \( u_{n+1} \) to compute the next \( y \)-value and use the resulting value to correct itself. Hence, the **corrector** is

\[
y_{n+1} = y_n + h \cdot \left( \frac{f(x_n, y_n) + f(x_{n+1}, u_{n+1})}{2} \right).
\]

- Each step of the improved Euler scheme requires two evaluations of the function \( f(x, y) \) compared with the single function required for the ordinary Euler scheme.

Example

To make a comparison to the Euler’s method, consider the initial value problem

\[
\frac{dy}{dx} = x \cdot y \quad y(0) = 1.
\]
Method

Here \( f(x, y) = x \cdot y \). Therefore,

\[
\begin{align*}
k_1 &= f(x_n, y_n) = x_n y_n \\
u_{n+1} &= y_n + h k_1 = y_n + h (x_n y_n) \\
k_2 &= f(x_{n+1}, u_{n+1}) = x_n (y_n + h x_n y_n) \\
y_{n+1} &= y_n + h \frac{(k_1 + k_2)}{2}
\end{align*}
\]

\( n = 0, 1, 2, \ldots \)

8.11.2 The Error in the Improved Euler Method

The local truncation error for the improved Euler’s method is of \( O(h^3) \) and therefore, the global truncation error is of \( O(h^2) \).

8.12 RUNGE-KUTTA METHOD

8.12.1 Second Order Runge-Kutta Method

A more accurate numerical method for finding the solution to an initial value problem of the form

\[
\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0
\]

is the Runge-Kutta method. The second order Runge-Kutta method is obtained from using a Taylor polynomial that agrees to the order of degree 2. This is given below.

\[
y_{n+1} = y_n + \frac{h}{2} (\alpha k_1 + \beta k_2), \quad n = 0, 1, 2, \ldots
\]

where \( h \) is the step size and

\[
\begin{align*}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \gamma h, y_n + \delta k_1)
\end{align*}
\]

where

\[
\alpha + \beta = 1, \quad \beta \gamma = \frac{1}{2} \quad \text{and} \quad \beta \delta = \frac{1}{2}.
\]

The above system equations produces infinitely many solutions. Thus when \( \alpha = \beta = \frac{1}{2} \) and \( \gamma = \delta = 1 \), the second order Runge-Kutta scheme becomes the improved Euler’s (or Huen’s) scheme.
8.12.2 Fourth Order Runge-Kutta Method

A more accurate approximation to the solution to the initial value problem described in (*) is the fourth order Runge-Kutta iteration scheme:

\[ y_{n+1} = y_n + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4), \quad n = 0, 1, 2, \ldots \]

where

\[ k_1 = f(x_n, y_n), \]
\[ k_2 = f(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}), \]
\[ k_3 = f(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}), \]
\[ k_4 = f(x_n + h, y_n + hk_3) \]

and \( h \) is the step size.

8.12.3 The Error in the Fourth Order Runge-Kutta Method

The local truncation error in this method is of \( O(h^5) \). Therefore, the global truncation error is \( O(h^4) \).

Example

Using the previous example, that is,

\[ y' = xy, \quad y(0) = 1 \]

make a comparison between Euler’s method and the fourth order Runge-Kutta schemes.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( y_n(x_n) )</th>
<th>Rel % Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>1.0050</td>
<td>1.0050</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>1.0202</td>
<td>1.02020</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>1.0460</td>
<td>1.0463</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>1.0832</td>
<td>1.0833</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>1.1331</td>
<td>1.1332</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.6</td>
<td>1.1971</td>
<td>1.1972</td>
<td>0.01</td>
</tr>
<tr>
<td>7</td>
<td>0.7</td>
<td>1.2774</td>
<td>1.2776</td>
<td>0.02</td>
</tr>
<tr>
<td>8</td>
<td>0.8</td>
<td>1.3768</td>
<td>1.3771</td>
<td>0.03</td>
</tr>
<tr>
<td>9</td>
<td>0.9</td>
<td>1.4988</td>
<td>1.4993</td>
<td>0.04</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>1.6479</td>
<td>1.6487</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Comparing the above results we can see as numerical scheme leaves the given initial values the greater the relative percentage error of approximation to the solution to the initial value problem. However, taking smaller step lengths will give better approximation.

The fourth order Runge-Kutta global truncation error is of \( O(h^4) \). Therefore, in this example, the global error truncation error is \( (0.1)^4 = \).
8.13 MULTI-STEP METHODS

The Euler and Runge-Kutta methods are examples of single-step methods in that each successive value of $y_n$ are obtained from the preceding calculation of $y_n$. However, a multi-step method uses the values from several previous calculated values to obtain an approximation for $y_{n+1}$.

Multi-step methods have their advantages and disadvantages. These methods still require initial calculations using the single-step methods. Therefore, accuracy will be dependent on the initial single-step method used. In general, the multi-step methods will require far less calculations than the 4th order Runge-Kutta method. Further, if the evaluation of $f(x,y)$ is complicated, then the multi-step method will also lead to better efficiency. Below is one such multi-step method called the Adams-Bashforth/Adams-Moulton Method.

Given the initial value problem

$$y' = f(x, y), \quad y(x_0) = x_0$$

then the Adams-Bashforth scheme is

$$u_{n+1} = y_n + \frac{h}{24} (55k_1 - 59k_2 + 37k_3 - 9k_4)$$

where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_{n-1}, y_{n-1})$$
$$k_3 = f(x_{n-2}, y_{n-2})$$
$$k_4 = f(x_{n-3}, y_{n-3})$$

for $n \geq 3$. This scheme is called the **predictor** method.

The value of $u_{n+1}$ is then substituted into the Adams-Moulton corrector method

$$k_5 = f(x_{n+1}, u_{n+1})$$

$$y_{n+1} = y_n + \frac{h}{24} (9k_5 + 19k_1 - 5k_2 + k_3).$$

The local truncation error of the Adams-Bashforth method is of $O(h^5)$. Therefore, the initial values for $y_n$ for $n = 1, 2$ are usually calculated by a method of the same order, like the fourth order Runge-Kutta method.
8.14 SECOND ORDER INITIAL VALUE PROBLEMS

In this section, we will be looking at numerical methods for solving second order initial value problems using preceding techniques. The procedure is simple. We first transform the second order differential equation into 2 first order differential equations. Due to the fact that we have now reduced down to a set of first order differential equations, we simply apply one of the numerical techniques that we have learnt already.

8.14.1 Reduction to a System of First Order Differential Equations

Consider the second order initial value problem of the form
\[ \frac{d^2 y}{dx^2} = f(x, y, \frac{dy}{dx}) \]
subject to \( y(x_0) = y_0 \) and \( \frac{dy(x_0)}{dx} = y'_0 \).

Let
\[ y' = \frac{dy}{dx} = u \] (1)
then
\[ u' = f(x, y, \frac{dy}{dx}) \] (2)
with initially, \( y(x_0) = y_0 \) and \( u(x_0) = \frac{dy(x_0)}{dx} = y'_0 \).

Since (1) and (2) are now first order initial value problems then Euler’s or Runge-Kutta technique can be used.

Example

Use Euler’s method to obtain the approximate value of \( y(1.2) \) where \( y(x) \) is the solution of the initial value problem
\[ x^2 y'' - xy' + y = 0 \]
subject to the initial conditions \( y(1) = 3 \) and \( y'(1) = -1 \).

Method

Let
\[ y' = u \] (3)
then
\[ x^2 u' - xu + y = 0. \] (4)
Upon rewriting (3) we find that
\[ u' = f(x, y, u) = \frac{xu - y}{x^2}, \] (5)
Thus, applying Euler’s scheme to (3) and (5) we have the following iteration scheme of the form:

\[
y_{n+1} = y_n + hu_n \\
u_{n+1} = u_n + h f(x_n, y_n, u_n) \\
= u_n + h \left( \frac{x_n u_n - y_n}{x_n^2} \right) \quad n = 0, 1, 2, \ldots
\]

where \( x_0 = 1, \ y_0 = 3 \) and \( u_0 = -1 \).

Using \( h = 0.1 \). Let \( n = 0 \) then

\[
y_1 = y_0 + h(u_0) \\
= 3 + (0.1) \times (-1) \\
= 2.9
\]

and

\[
u_1 = u_0 + h \left( \frac{x_0 u_0 - y_0}{x_0^2} \right) \\
= -1 + (0.1) \times \left( \frac{1 \times (-1) - 3}{1^2} \right) \\
= -1.4.
\]

Hence, an approximation to the solution at \( x = 1.1 \) is \( y_1 = 2.9 \). That is, \( y(1.1) \approx 2.9 \).

Let \( n = 1 \) then

\[
y_2 = y_1 + h(u_1) \\
= 2.9 + (0.1) \times (-1.4) \\
= 1.74
\]

and

\[
u_2 = u_1 + h \left( \frac{x_1 u_1 - y_1}{x_1^2} \right) \\
= -1.4 + (0.1) \times \left( \frac{1.1 \times (-1.4) - 2.9}{1.1^2} \right) \\
= -1.77.
\]

This calculation is corrected to two decimal places (2 d.p.). Thus, under this correction, calculation errors are introduced. Hence, an approximation to the solution at \( x = 1.2 \) is \( y_2 = 1.74 \). That is, \( y(1.2) \approx 1.74 \).

**Exercise 8C**

1. Use Newton’s method to find the root of the function

\[
f(x) = 8x^3 + 12x^2 + 130x + 63
\]

closest to \( 0 \).

Note that if a close approximation to the root of (*) is not given, a reasonable estimate of the root can be obtained by plotting \( f(x) \)

2. Given the initial value problem

\[
\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1.
\]

(a) Use Euler’s method to approximate the solution to the given initial value problem on the interval \([0, 1]\) using steplengths of \( h = 0.1, \ h = 0.02 \) and \( h = 0.005 \).

Compare to the analytic solution.

Comment on your result(s).
(b) Use the fourth order Runge-Kutta method with steplength of \( h = 0.1 \) to approximate the solution to the given initial value problem on the interval \([0, 1]\).

Compare results to the that obtained in (a).

3 Use Euler’s method to obtain an approximate solution of \( y(0.2) \) where \( y(x) \) is a solution of the initial value problem

\[
\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0
\]

subject to \( y(0) = 1, \ \frac{dy}{dx} = 2 \).

### 8.15 SECOND ORDER BOUNDARY VALUE PROBLEMS

#### 8.15.1 Introduction

A variety of boundary-values problems can be solved analytically or symbolically. However, many differential equations and boundary value problems cannot be solved using known techniques. Therefore, if an analytical approach does not yield a solution then we may be able to resort to numerical techniques.

One of the most common numerical techniques that is used to solve boundary value problems is the use of finite difference approximations. This will be discussed in this section.

**Example**

Solve the following boundary value problem

\[
\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = g(x), \quad 0 < x < 1
\]

where \( y(0) = a \) and \( y(1) = b \).

**Note**

This equation is a second order non-homogeneous constant coefficient differential equation with non-homogeneous boundary conditions.

We can use symbolic software or analytic techniques to solve such an equation. (See MATH202 - Analytic part). However, when analytic solutions cannot be found then we must resort to numerical methods.

#### 8.15.2 Finite Difference Method

The central difference approximations of \( O(h^2) \) for the first and second derivative of the function \( y(x) \) are given by:

\[
\begin{align*}
\frac{dy}{dx} &= \frac{f(x + h) - f(x - h)}{2h} \\
\frac{d^2 y}{dx^2} &= \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}
\end{align*}
\]

(\( ** \))

where \( h \) is the step length between two successive \( x \) values.
Let \( x \) be defined on the interval \([a, b]\) where the interval is subdivided into \( n \) equal subintervals so that

\[
a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.
\]

Therefore, \( x_i = a + ih \) where \( i = 0, 1, 2, \ldots, n \) and \( h = (b - a)/n \). The points \( x_1, x_2, x_3, \ldots, x_{n-1} \) are called **interior mesh points** of the interval \([a, b]\). Using a finite difference scheme, the function values at these **interior points** (that is, \( y_1, y_2, y_3, \ldots, y_{n-1} \)) are to be determined. The **exterior points** \( x_0 \) and \( x_n \) are the known boundary points, that is, \( x_0 = a \) and \( x_n = b \).

Let

\[
y_i = y(x_i), \quad f_i = f(x_i) \quad \text{and} \quad g_i = g(x_i).
\]

then the central difference approximations in (**) becomes

\[
y'_i = \frac{y_{i+1} - y_{i-1}}{2h},
\]

\[
y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.
\]

Substituting these approximation into equation (*) and simplifying we have a system of equations that we need to solve, namely,

\[
\left(1 + \frac{h}{2}\right)y_{i+1} + (h^2 - 2)y_i + \left(1 - \frac{h}{2}\right)y_{i-1} = h^2 g_i,
\]

where \( i = 1, 2, 3 \ldots, n-1 \).

**Note**

Equation (***) is called the finite difference equation which is an approximation to the given differential equation in (*). Also, both \( y_0 \) and \( y_n \) are known as they are determined by the prescribed boundary conditions associated with (*).

8.15.3 A System of Linear Differential Equations

Several methods are available to solve systems of differential equations, in particular, linear system of equations. This can also be done symbolically. However, the symbolic solutions do not give enough information about the system of differential equations compared to an analytic one. We shall look at one such analytic method in this section. This method uses matrix algebra to decouple the system of differential equations through spectrally decomposing a matrix.

**Example**

Consider the system of linear equations:

\[
\frac{d^2 x_1}{dt^2} = -2x_1 + x_2,
\]

\[
\frac{d^2 x_2}{dt^2} = x_1 - 2x_2
\]
This system of equations can be written in matrix form. That is,

$$\ddot{X} = -A \dot{X}$$  \hspace{1cm} (1)

where $X$ is a vector made up of the unknowns and $A$ is the matrix of coefficients. That is,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \dot{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The spectral decomposition of $A$ is obtained by first finding the eigenvalues and then the eigenvectors of the matrix $A$. The eigenvalues of $A$ are found by letting

$$|A - \lambda I| = 0.$$

For this example, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Thus the corresponding eigenvectors are, respectively, $(1,1)^T$ and $(1,-1)^T$. Hence,

$$A = PDP^{-1}$$

is the spectral decomposition of $A$ where $D$ is a diagonal matrix made up of eigenvalues of $A$ and $P$ is the matrix made up of the corresponding eigenvectors.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Substituting $\dot{Z} = P^{-1} \dot{X}$ into (1), we have a decoupled system of differential equations of the form

$$\ddot{Z} = -D \dot{Z}. \hspace{1cm} (2)$$

Since $D$ is a diagonal matrix, (2) can be readily solved by using the techniques learnt in MATH203. That is,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} C_1 \cos(\sqrt{\lambda_1} t + \epsilon_1) \\ C_2 \cos(\sqrt{\lambda_2} t + \epsilon_2) \end{pmatrix} = \begin{pmatrix} C_1 \cos(t + \epsilon_1) \\ C_2 \cos(3t + \epsilon_2) \end{pmatrix} \hspace{1cm} (3)$$

where $\dot{z} = (z_1, z_2)^T$. Once the solution $\dot{Z}$ is found, $\dot{X}$ can then be obtained by using the fact that

$$\ddot{X} = P \ddot{Z}.$$

Thus, we only need to find the eigenvalues and eigenvectors of the system of differential equations. Matlab can help us here.
Exercise 8D

1. Find the root closest to 1 for
\[ \frac{x^3}{8} - x + 1 = 0 \]
and closest to 0 for
\[ x^3 - 6x^2 + 11x - 6 = 0. \]

Use the Newton-Raphson scheme and the iterative scheme \( x_{n+1} = g(x_n) \) (after rewriting the equations in the form \( x = g(x) \)) to find the roots correct to 3 decimal places.

2. Consider the problem of finding the smallest non zero root of \( x - 2 \sin 2x = 0 \). Which of the following rearrangements (\( x = g(x) \)) should you choose
   (i) \( x = 2 \sin 2x \), or
   (ii) \( x = \frac{\pi}{2} - \frac{1}{2} \sin^{-1} \left( \frac{x}{2} \right) \).

Give reasons for your choice and find the and find the root correct to 3 decimal places.

3. Use the Newton-Raphson scheme to find the root(s) of \( x - e^{-x} = 0 \) correct to 3 decimal places.

4. Use a difference approximation to evaluate \( \frac{df}{dx} \) and \( \frac{d^2f}{dx^2} \) at \( x = 1 \), correct to \( O(h^2) \),
   (a) \( f(x) = \cos \left( \frac{x}{1 + x^2} \right) \)
   (b) \( f(x) = \exp(x^3 + x) \)
when
   (i) step length \( h = 0.1 \) and
   (ii) step length \( h = 0.01 \).

5. Evaluate
   (a) \( \cos(x) \) and (b) \( \sin(x^2) \)
at points \(-0.5, 0 \) and \(0.5 \).

Use these values to produce quadratic approximations to both functions.

What is the error of the approximations at points \(0.25 \) and \(1 \)?

6. Consider the following integrals
   (a) \( \int_0^1 \cos(x^2)\,dx \) and
   (b) \( \int_{-1}^1 \frac{dx}{1 + x^2} \)
by means of
   (i) the trapezoidal rule with \([0, 1]\) divided into 4 intervals (\( h = 0.25 \)),
   (ii) the trapezoidal rule with \([0, 1]\) divided into 8 intervals (\( h = 0.125 \)),
   (iii) Simpsons rule with \([0, 1]\) divided into 4 intervals,
   (iv) Simpsons rule with \([0, 1]\) divided into 8 intervals,

7. Consider the ODE
   \[ \frac{dy}{dx} = \frac{1}{1 + y} \] subject to \( y(0) = 0 \).

Use the Euler formula with step length \( h = 0.1 \) to find \( y \) at point 1.

Repeat the calculation with step length \( h = 0.05 \).

8. Find \( y(0.4) \) for \( y(x) \) satisfying
   \[ \frac{dy}{dx} = (1 - y^2)^{\frac{1}{2}}, \]
subject to \( y(0) = 0 \).
   (a) Use an Euler scheme with step length 0.1 and then a 2nd order Runge-Kutta scheme with step length 0.1.
   (b) Compare your answers in 1(b) with those obtained by using a 4th order Runge-Kutta scheme with first a step length of 0.4 and then a step length of 0.2

continued next page...
9. Use the 2D Newton-Raphson scheme to find the root closest to \((1, 1)\) for the system
\[
\begin{pmatrix}
15x + x^2 + y^2 - 15 \\
10y + x^2 - 10
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Find root correct to 3 decimal places.

10. Use the 2D Newton-Raphson scheme to find the root closest to \((0, 0)\) for the system
\[
\begin{pmatrix}
x^2 - \sin(x + y) - 1 \\
y - \cos(x + y)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Find root correct to 2 decimal places.

11. Find \(y(x)\) in \([0, 1]\) for \(\frac{d^2 y}{dx^2} = y + x^2\) subject to \(y(0) = y(1) = 0\).
Use a finite difference scheme in which \([0, 1]\) is divided into 3 equal length intervals. Then, repeat the calculation with \([0, 1]\) divided into 5 equal length intervals.
8.16 NUMERICAL CODING

The following Matlab codings are for MATH202 tutorial and assignment problems in 2001 which may be used this year. These codings are usually self explanatory. Further detail will be given during the delivery of the numerical section of MATH202.

8.16.1 Taylor Polynomials

% Approximate y = exp(x) using Taylor Polynomials of degree 0, 1, 2, and 3.
% Define region of interest as [x0, xn]
% Divide region into n sub-regions using stepsize h = (xn - x0)/n
x0 = 0;
xn = 1;
n = 10;
h = (xn - x0)/n;
% Calculate x, y(x), p0(x), p1(x), p2(x), and p3(x)
% Calculate absolute error = exact value - approximate value
% Calculate relative error = absolute error / exact value * 100
for i = 1:n+1;
x(i) = h*(i-1); % calc. x values
y(i) = exp(x(i)); % calc. y values
p0(i) = 1; % calc. poly. approx. of degree 0
abserr0(i) = y(i) - p0(i); % absolute error using p0
relerr0(i) = 100*abserr0(i)/y(i); % relative error using p0
p1(i) = p0(i) + x(i); % calc. poly. approx. of degree 1
abserr1(i) = y(i) - p1(i); % absolute error using p1
relerr1(i) = 100*abserr1(i)/y(i); % relative error using p1
p2(i) = p1(i) + x(i)^2/2; % calc. poly. approx. of degree 2
abserr2(i) = y(i) - p2(i); % absolute error using p2
relerr2(i) = 100*abserr2(i)/y(i); % relative error using p2
p3(i) = p2(i) + x(i)^3/6; % calc. poly. approx. of degree 3
abserr3(i) = y(i) - p3(i); % absolute error using p3
relerr3(i) = 100*abserr3(i)/y(i); % relative error using p3
end;
% Plot exact and approximate solutions
figure(1); % open figure no.1 window
plot(x,p0,x,p1,x,p2,x,p3,x,y); % generate figure no. 1
legend('y=exp(x)', 'y=p0(x)', 'y=p1(x)', 'y=p2(x)', 'y=p3(x)'); % add legend
title('Exact solution and Taylor Polynomial approximations'); % add title
xlabel('X values'); % add label to x axis
ylabel('Y values'); % add label to y axis
% Plot absolute errors
figure(2); % open figure no. 2 window
plot(x,abserr0,x,abserr1,x,abserr2,x,abserr3); % generate figure no. 2
legend('using p0(x)', 'using p1(x)', 'using p2(x)', 'using p3(x)'); % add legend to figure

% Add title

title('Absolute Errors for Taylor Polynomial approximations'); % add title

% Add label to x axis

xlabel('X values'); % add label to x axis

% Add label to y axis

ylabel('Absolute error values'); % add label to y axis

figure(3); % open figure no. 3 window

plot(x, relerr0, x, relerr1, x, relerr2, x, relerr3); % generate figure no. 3

legend('y=exp(x)', 'y=p0(x)', 'y=p1(x)', 'y=p2(x)', 'y=p3(x)'); % add legend to figure

% Add title

title('Relative Errors for Taylor Polynomial approximations'); % add title

% Add label to x axis

xlabel('X values'); % add label to x axis

% Add label to y axis

ylabel('Relative (Percentage) error values'); % add label to y axis
8.16.2 Numerical Integration

% Evaluate definite integral of y = f(x) for x in [a, b] using three simple quadrature methods:
% Midpoint rule integral = (b-a)*f((a+b)/2)
% Trapezoidal rule integral = (b-a)*(f(a)+f(b))/2
% Simpson’s rule integral = (b-a)*(f(a)+4*f((a+b)/2)+f(b))/6
% Evaluate absolute error in quadrature = exact value - approximation
% Evaluate relative error in quadrature = 100 * absolute error / exact value
% Plot y = f(x), quadrature results, and errors in quadrature results
% Define region of interest as [a, b]
   a = 0;
   b = 2;
% Calculate exact value of integral of f(x) = x^2 for x in [a, b]
   exact = (bˆ3 - aˆ3) / 3 ;
% For all three quadrature methods, we normally divide region into n sub-regions
% using stepsize of h = (b - a) / n, approximate the integral in each sub-region using
% the above simple quadrature formulae, and then sum these results.
% Define maximum number of sub-intervals
   maxint = 16;
% Repeat calculations for between 1 and max of sub-intervals
   for n = 1:maxint;
% We need to evaluate (x,y) at the midpoint and both ends of each sub-region.
% Thus for n sub-regions we need to evaluate 2*n+1 ordered pairs (x,y).
% for example: for x in [a, b] and n = 1, we need to evaluate three ordered
% pairs (a,f(a)), ((a+b)/2,f((a+b)/2)), and (b,f(b)), which correspond to
% (x(1),y(1)), (x(2),y(2)), and (x(3),y(3)) respectively.
   numint(n) = n; % number of sub-regions
   h = (b - a)/n; % width of each sub-region
% Calculate y = f(x) for each ordered pair
   for j = 1:2*n+1;
      x(j) = a + h*(j-1)/2; % calc. x
      y(j) = x(j)^2; % calc. y = f(x) = x^2
   end;
% Approximate definite integral using midpoint rule.
midp(n) = 0;
for j = 2:2:2*n;
    midp(n) = midp(n) + y(j);
end;
midp(n) = midp(n) * h;
abserrm(n) = exact - midp(n);
relerrm(n) = 100*abserrm(n)/exact;

% Approximate definite integral using trapezoidal rule.
trap(n) = (y(1) + y(2*n+1))/2;
for j = 3:2:2*n-1;
    trap(n) = trap(n) + y(j);
end;
trap(n) = trap(n) * h;
abserrt(n) = exact - trap(n);
relerrt(n) = 100*abserrt(n)/exact;

% Approximate definite integral using Simpsons rule.
simp(n) = y(1) + y(2*n+1);
for j = 2:2*n;
    if mod(j,2) == 0
        simp(n) = simp(n) + 4*y(j);
    else
        simp(n) = simp(n) + 2*y(j);
    end;
end;
simp(n) = simp(n) * h / 6;
abserrs(n) = exact - simp(n);
relerrs(n) = 100*abserrs(n)/exact;
end;
% Plot function \( y = f(x) \)

```matlab
figure(1);
plot(x,y); title('Graph of \( y = f(x) \)');
xlabel('X values');
ylabel('Y values');
```

% Plot quadrature results

```matlab
figure(2); plot(numint,midp,numint,trap,numint,simp);
legend('Midpoint rule, Trapezoidal rule','Simpsons rule');
title('Quadrature results for \( y = f(x) \)');
xlabel('Number of Intervals');
ylabel('Quadrature Result');
```

% Plot absolute errors in quadrature results

```matlab
figure(3); plot(numint,abserrm,numint,abserrt,numint,abserrs);
legend('Midpoint rule', 'Trapezoidal rule', 'Simpsons rule');
title('Absolute errors in quadrature results for \( y = f(x) \)');
xlabel('Number of Intervals');
ylabel('Absolute error values');
```

% Plot relative errors in quadrature results

```matlab
figure(4); plot(numint,relerrm,numint,relerrt,numint,relerrs);
legend('Midpoint rule', 'Trapezoidal rule', 'Simpsons rule');
title('Relative errors in quadrature results for \( y = f(x) \)');
xlabel('Number of Intervals');
ylabel('Relative error values');
```
8.16.2 Euler’s Method

% Approximate the solution of the initial value problem:
% y' = f(x, y) for x in [x0, xn] and y(x0) = y0
% at n+1 equally spaced points in the interval [x0, xn]
% using Eulers method.
% Define region of interest as [x0, xn]
x0 = 0;
xn = 1;
% Define initial condition as y0 = y(x0)
y0 = 1;
% Divide region into n sub-regions using stepsize h = (xn - x0)/n
n = 2;
h = (xn - x0)/n;
% Calculate y(x) using Eulers method.
x(1) = x0;
yeuler(1) = y0;
for i=1:n;
f = 2*x(i) + yeuler(i); % calc. f(x, y)
x(i+1) = x(i) + h; % calc. next x
yeuler(i+1) = yeuler(i) + h * f; % calc. next y(x)
end;
% Calculate exact value using 3*exp(x)-2*(x+1).
% Calculate absolute error = exact value - approximate value
% Calculate relative error = 100* absolute error / exact value
for i = 1:n+1;
yexact(i) = 3*exp(x(i))-2*(x(i)+1) % calc. exact solution
abserr(i) = abs(yexact(i)-yeuler(i)); % calc. absolute error
relerr(i) = 100*abserr(i)/yexact(i); % calc. relative error
end;
% Plot exact and approximate solutions
figure(1);
plot(x,yeuler,x,yexact);
legend('Euler','Exact');
title('Solutions to initial value problem Dy = f(x, y)');
xlabel('X values');
ylabel('Y values');
figure(2);
plot(x,abserr);
title('Absolute Errors using Euler approximation');
xlabel('X values');
ylabel('Absolute error values');
% Plot relative errors
figure(3);
plot(x,relerr);
title('Relative Errors using Euler approximation');
xlabel('X values');
ylabel('Relative error values');
8.16.3 Huen’s Method

% Approximate the solution of the initial value problem:
% y' = f(x, y) for x in [x0, xn] and y(x0) = y0
% at n+1 equally spaced points in the interval [x0, xn]
% using Eulers and Huen's method.
% Initialise workspace

clear;
% Define region of interest as [x0, xn]
x0 = 0;
xn = 3;
% Define initial condition as y0 = y(x0)
y0 = 1;
% Divide region into n sub-regions using stepsize h = (xn - x0)/n
n = 10;
h = (xn - x0)/n;
% Calculate y(x) using Eulers and Huen's methods.
x(1) = x0;
yeuler(1) = y0;
yhuen(1) = y0;
for i=1:n;
x(i+1) = x(i) + h; % calc. next x
feuler = 2*x(i) + yeuler(i); % calc. f(x, y)
yeuler(i+1) = yeuler(i) + h * feuler; % calc. next y(x)
fhuen1 = 2*x(i) + yhuen(i);
ypredictor = yhuen(i) + h * fhuen1; % calc. y prediction
fhuen2 = 2*x(i+1) + ypredictor;
yhuen(i+1) = yhuen(i) + h/2 * (fhuen1 + fhuen2); % calc. y correction
end;
% Calculate exact value using 3*exp(x)-2*(x+1).
% Calculate absolute error = exact value - approximate value
% Calculate relative error = 100* absolute error / exact value
for i = 1:n+1;
yexact(i) = 3*exp(x(i))-2*(x(i)+1); % calc. exact solution
abserre(i) = abs(yexact(i)-yeuler(i)); % calc. absolute error in Euler
relerre(i) = 100*abserre(i)/yexact(i); % calc. relative error in Euler
abserrh(i) = abs(yexact(i)-yhuen(i)); % calc. absolute error in Huen
relerrh(i) = 100*abserrh(i)/yexact(i); % calc. relative error in Huen
end;
% Plot exact and approximate solutions
figure(1);
plot(x,yeuler,x,yhuen,x,yexact);
legend('Euler','Huen','Exact');
title('Solutions to initial value problem Dy = f(x, y)');
xlabel('X values');
ylabel('Y values');
% Plot absolute errors
figure(2);
plot(x,abserre,x,abserrh);
legend('Euler','Huen');
title('Absolute Errors using Euler and Huen approximations');
xlabel('X values');
ylabel('Absolute error values');

% Plot relative errors
figure(3);
plot(x,relerre,x,relerrh);
legend('Euler','Huen');
title('Relative Errors using Euler and Huen approximations');
xlabel('X values');
ylabel('Relative error values');
8.16.4 M-files

File Name: MainProg.m

% Approximate the solution of the initial value problem:
% y' = f(x, y) for x in [x0, xn] and y(x0) = y0
% at n+1 equally spaced points in the interval [x0, xn]
% using Euler, Huen,
% stored as function m-files euler and huen.
% Define region of interest as [x0, xn]
x0 = 0;
xn = 1;
% Define initial condition as y0 = y(x0)
y0 = 0;
% Divide region into n sub-regions using stepsize h = (xn - x0)/n
n = 10;
h = (xn - x0)/n;
% Calculate y(x) using Euler, Huen, and Runge-Kutta methods.
x(1) = x0;
yeuler(1) = y0;
yhuen(1) = y0;
for i=1:n;
x(i+1) = x(i) + h; % calc. next x
yeuler(i+1) = euler(x(i),yeuler(i),h); % calc. next y(x) using Euler
yhuen(i+1) = huen(x(i),yhuen(i),h); % calc. next y(x) using Huen
end;
% Calculate exact value using 2/3x-2/9+2/9e^{-3x} .
% Calculate absolute error = exact value - approximate value
% Calculate relative error = 100* absolute error / exact value
for i = 1:n+1;
yexact(i) = 2/3*x(i)-2/9+2/9*exp(-3*x(i)); % calc. exact solution
abserre(i) = abs(yexact(i)-yeuler(i)); % calc. absolute error euler
relerre(i) = 100*abserre(i)/yexact(i); % calc. relative error euler
abserrh(i) = abs(yexact(i)-yhuen(i)); % calc. absolute error huen
relerrh(i) = 100*abserrh(i)/yexact(i); % calc. relative error huen
end;
% Plot exact and approximate solutions
figure(1);
plot(x,yeuler,x,yhuen,x,yexact);
legend('Euler','Huen','Exact');
title('Solutions to initial value problem Dy = f(x, y)');
xlabel('X values');
ylabel('Y values');
% Plot absolute errors
figure(2);
plot(x,abserre,x,abserrh);
legend('Euler','Huen');
title('Absolute Errors using various approximations');
xlabel('X values');
ylabel('Absolute error values');
% Plot relative errors
figure(3);
plot(x,relerre,x,relerrh);
legend('Euler','Huen');
title('Relative Errors using various approximations');
xlabel('X values');
ylabel('Relative error values');
figure(4);
plot(x,yeuler,x,yhuen,x,yexact);
legend('Euler','Huen','Exact');
title('Solutions to initial value problem Dy = f(x, y)');
xlabel('X values');
ylabel('Y values');

FileName: f202.m

% Approximate the solution of the initial value problem:
% y' = f(x, y) for x in [x0, xn] and y(x0) = y0
% at n+1 equally spaced points in the interval [x0, xn]
% using various method.
% this m-file defines the function f202(x,y,h):
% it calculates the next iteration of y using the euler method.
% input parameters are x, y, and h
% output is ynext
% note: function name must be the same as the file name
function [fxy]=f202(x,y)
fxy = 2 * x - 3 * y;
FileName: euler.m

% Approximate the solution of the initial value problem:
% y' = f(x, y) for x in [x0, xn] and y(x0) = y0
% at n+1 equally spaced points in the interval [x0, xn]
% using various method.
% this m-file defines the function euler(x,y,h):
% it calculates the next iteration of y using the euler method.
% input parameters are x, y, and h
% output is ynext
% note: function name must be the same as the file name
% note: references m-file f202
function [ynext]=euler(x,y,h)
    f1 = f202(x,y);
    ynext = y + h*f1;

FileName: huen.m

% Approximate the solution of the initial value problem:
% y' = f(x, y) for x in [x0, xn] and y(x0) = y0
% at n+1 equally spaced points in the interval [x0, xn]
% using various method.
% this m-file defines the function huen(x,y,h):
% it calculates the next iteration of y using the huen method.
% input parameters are x, y, and h
% output is ynext
% note: function name must be the same as the file name
% note: references m-file f202
function [ynext]=huen(x,y,h)
    f1 = f202(x,y);
    ynext = y + h*(y + h*f1);
    yprediction = y + h*f1;
    f2 = f202(x+h,yprediction);
    ynext = y + h/2*(f1+f2);
8.16.3 Runge-Kutta System of Equations

% y' = f(x, y) for x in [x0, xn] and y(x0) = y0
x0 = 1;
xn = 3;
y0 = 5;
n = 20;
h = (xn - x0)/n;
x(1) = x0;
yeuler(1) = y0;
yhuen(1) = y0;
ymid(1) = y0;
yrunge(1) = y0;
for i=1:n;
x(i+1) = x(i) + h;
yeuler(i+1) = euler(x(i),yeuler(i),h);
yhuen(i+1) = huen(x(i),yhuen(i),h);
ymid(i+1) = midpt(x(i),ymid(i),h);
yrunge(i+1) = rk(x(i),yrunge(i),h);
end;
for i = 1:n+1;
yexact(i) = 38/9 * exp(3*(1-x(i)))+2/3*x(i)+1/9
abserre(i) = abs(yexact(i)-yeuler(i));
relerre(i) = 100*abserre(i)/yexact(i);
abserrh(i) = abs(yexact(i)-yhuen(i));
relerrh(i) = 100*abserrh(i)/yexact(i);
abserrm(i) = abs(yexact(i)-ymid(i));
relerrm(i) = 100*abserrm(i)/yexact(i);
abserrr(i) = abs(yexact(i)-yrunge(i));
relerrr(i) = 100*abserrr(i)/yexact(i);
end;
figure(1);
plot(x,yeuler,x,yhuen,x,yrunge,x,yexact);
legend('Euler','Huen','4Runge','Exact');
title('Solutions to initial value problem Dy = f(x, y)');
xlabel('X values');
ylabel('Y values');
figure(2);
plot(x,abserre,x,abserrh,x,abserrr);
legend('Euler','Huen','4Runge');
title('Absolute Errors using various approximations');
xlabel('X values');
ylabel('Absolute error values');
figure(3);
plot(x,relerre,x,relerrh,x,relerrr);
legend('Euler','Huen','4Runge');
title('Relative Errors using various approximations');
xlabel('X values');
ylabel('Relative error values');
figure(4);
plot(x,yeuler,x,yhuen,x,yrunge,x,yexact);
legend('Euler','Huen','4Runge','Exact');
title('Solutions to initial value problem Dy = f(x, y)');
xlabel('X values');
ylabel('Y values');
hold on;
plot(x,ymid,`–rs`,'MarkerFaceColor','m');
hold off;
figure(5);
plot(x,abserre,x,abserrh,x,abserrr);
legend('Euler','Huen','4Runge');
title('Absolute Errors using various approximations');
xlabel('X values');
ylabel('Absolute error values');
hold on;
plot(x,abserrm,`–rs`,'MarkerFaceColor','m');
hold off;
figure(6);
plot(x,relerre,x,relerrh,x,relerrr);
legend('Euler','Huen','4Runge');
title('Relative Errors using various approximations');
xlabel('X values');
ylabel('Relative error values');
hold on;
plot(x,relerrm,`–rs`,'MarkerFaceColor','m');
hold off;
figure(7);
plot(x,yhuen);
legend('Huen');
title('Solutions to initial value problem Dy = f(x, y)');
xlabel('X values');
ylabel('Y values');
hold on;
plot(x,ymid,`–rs`,'MarkerFaceColor','m');
hold off;
8.17 MATHEMATICA CODING

The following is a list of *Mathematica* coding for the Euler, Huen and fourth order Runge-kutta methods.

8.17.1 Euler’s Method

```mathematica
Clear[f,h,x,y,yi,yvalues,truey,xypoints]
h=0.1;
f[x_,y_] := x y;
y[0]=1;
x[n_]= n h;
y[n_]:=Module[{k1,k2},
k1= h f[x[n-1],y[n-1]];
y[n]=y[n-1]+ k1
]
yvalues=Table[y[i],{i,0,10}];
truey[i]=Exp[x[i]^2];
errori=Table[Abs[truey[i]-y[i]]/truey[i],{i,0,10}];
ytrue = Table[truey[i],{i,0,20}];
xyvalues=Table[{n,x[n-1],yvalues[[n]],ytrue[[n]],error[[n]]},{n,1,Length[yvalues]}]
]/TableForm
```

8.17.2 Modified Euler’s Method

```mathematica
Clear[f,h,x,y,yi,yvalues,truey,xypoints]
h=0.1;
f[x_,y_] := x y;
y[0]=1;
x[n_]= n h;
y[n_]:=Module[{k1,k2},
k1= h f[x[n-1],y[n-1]];
k2= h f[x[n-1],y[n-1]+ k1];
y[n]=y[n-1]+ (k1+k2)/2
]
yvalues = Table[y[i], {i, 0, 10}];
truey[i] := Exp[x[i]^2];
errori = Table[Abs[truey[i] - y[i]]/truey[i], {i, 0, 10}];
ytrue = Table[truey[i], {i, 0, 20}];
Table[{n, x[n-1], yvalues[[n]], ytrue[[n]], error[[n]]}, {n, 1, Length[yvalues]}] // TableForm

8.17.3 Fourth Order Runge-Kutta Method

Clear[f, h, x, y, y[i], yvalues, truey, xypoints]
h = 0.1;
f[x_, y_] := x y;
y[0] = 1;
x[n_] := n h;
y[n_] := Module[{k1, k2, k3, k4},
  k1 = h f[x[n-1], y[n-1]];
  k2 = h f[x[n-1] + h/2, y[n-1] + k1/h];
  k3 = h f[x[n-1] + h/2, y[n-1] + k2/h];
  k4 = h f[x[n-1] + h, y[n-1] + k3];
  y[n] = y[n-1] + (k1 + 2 k2 + 2 k3 + k4)/6
]
yvalues = Table[y[i], {i, 0, 10}];
truey[i] := Exp[x[i]^2];
errori = Table[Abs[truey[i] - y[i]]/truey[i], {i, 0, 10}];
ytrue = Table[truey[i], {i, 0, 20}];
xyvalues = Table[{n, x[n-1], yvalues[[n]], ytrue[[n]], error[[n]]}, {n, 1, Length[yvalues]}] // TableForm
Chapter 10: Tests and Exam Papers
MATH283 - DIFFERENTIAL EQUATIONS

Multiple Choice Test Sample - 1999

Duration - 55 minutes.

Number of questions: 9

All questions of equal value.

Non-alphanumeric calculators with single-line display permitted.

TEST PAPER MUST BE HANDED BACK
WITH COMPUTER SHEET ENCLOSED.

Make sure that the Test Paper and Computer Sheet are filled out correctly.

Failure to do so may lead to zero marks for the Multiple Choice Test.

1. Let \( z = f(u) \), where \( u = \tan(2x + 3y) \). \( \frac{\partial z}{\partial y} \) in terms of \( u \) is
   
   (a) \( f'(u)u^2 \)  (b) \( 3f'(u)(1 + u^2) \)  (c) \( 2f'(u)(1 + u^2) \)  (d) \( 3f'(u) \)  (e) \( 3f'(u)(1 - u^2) \).

2. A particular integral for the second order differential equation

   \( y'' + 2y' + y = -4e^{-x} \sin 2x \)

   is

   (a) \( (A \cos 2x + B \sin 2x)e^{-x} \)  (b) \( (\cos x + \sin x)e^{2x} \)  (c) \( e^{-x} \cos 2x \)
   (d) \( e^{-x}(\cos 2x + \sin 2x) \)  (e) \( e^{-x} \sin 2x \).
3. Let \( R \) be the region bounded by the curves \( y = \sin x \) and \( y = \cos x \) for \( x \) lying between \( x = 0 \) and \( x = \frac{\pi}{4} \). The value of
\[
\int \int_R ye^x \, dy \, dx
\]
is
(a) \( \frac{2 - e^{\pi/4}}{4} \) \hspace{1cm} (b) \( 2 - e^{\pi/4} \) \hspace{1cm} (c) \( \frac{e^{\pi/4}}{\sqrt{2}} - 1 \) \hspace{1cm} (d) \( \frac{2e^{\pi/4} - 1}{5} \) \hspace{1cm} (e) \( \frac{1}{10}(2e^{\pi/4} - 1) \).

4. Evaluate the total derivative \( dz \) when
\[
z = x^2 - 3xy + 2y^2 \quad \text{at} \quad x = 2, \ y = -3, \ \Delta x = -0.3, \ \Delta y = 0.2.
\]
(a) 3.4 \hspace{1cm} (b) 0.3 \hspace{1cm} (c) -2.3 \hspace{1cm} (d) -3.9 \hspace{1cm} (e) -7.5.

5. Find the second order mixed partial derivative \( \frac{\partial^2 z}{\partial x \partial y} \) of the function \( z = f(x, y) \) given by
\[
z = (x^2 + y^2) \ln(xy).
\]
(a) \( \frac{2(x^2 + y^2)}{xy} \) \hspace{1cm} (b) \( 2x \ln(xy) \) \hspace{1cm} (c) \( 2x \ln(xy) + \frac{x^2 + y^2}{x} \)
\hspace{1cm} (d) \( \frac{y^2 + 2xy - x^2}{y^2} \) \hspace{1cm} (e) \( x \ln x + y \ln y \).

6. Evaluate \( \int_0^\infty x^4 e^{-x^4} \, dx \).
(a) \( \frac{\Gamma\left(\frac{5}{4}\right)}{16} \) \hspace{1cm} (b) \( \frac{\sqrt{\pi}}{2} \{\text{erf}(4) - \text{erf}(3)\} \) \hspace{1cm} (c) \( \frac{\sqrt{\pi}}{2} \text{erf}(4) \) \hspace{1cm} (d) \( \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma(5)} \) \hspace{1cm} (e) \( \frac{\sqrt{\pi}}{2} \).

7. Evaluate \( \int_0^1 e^{-(u+1)^2} \, du \).
(a) \( \frac{\Gamma\left(\frac{3}{4}\right)}{2} \) \hspace{1cm} (b) \( \frac{\sqrt{\pi}}{2} \{\text{erf}(2) - \text{erf}(1)\} \) \hspace{1cm} (c) \( \frac{\sqrt{\pi}}{2} \text{erf}(1) \) \hspace{1cm} (d) \( \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma(4)} \) \hspace{1cm} (e) \( \frac{\sqrt{\pi}}{2} \).

8. Let \( Y(s) = \mathcal{L}\{y(x)\} \). If
\[
y'(x) + y(x) = h(x - 2)e^{-x} \quad \text{where} \quad y(0) = 1,
\]
then the Laplace transform \( Y(s) \) is
(a) \( \frac{1}{s - 1} + \frac{e^{2-s}}{(s+1)} \) \hspace{1cm} (b) \( \frac{e^{-s}}{(s+1)^2} \) \hspace{1cm} (c) \( \frac{-1}{s+1} + \frac{e^{-(s+2)}}{(s^2 - 1)} \)
\hspace{1cm} (d) \( \frac{1}{s+1} + \frac{e^{-2(s+1)}}{(s+1)^2} \) \hspace{1cm} (e) \( \frac{1}{s+1} + \frac{e^{-2s}}{(s+1)(s+2)} \).
9. Find the Laplace transform of the function $f(t) = t \cosh 2t$.

(a) \[\frac{4 - 2s + s^2}{(s^2 - 4)^2}\]  
(b) \[\frac{1}{4 - s^2}\]  
(c) \[\frac{s^2 + 4}{(s - 2)^2(s + 2)^2}\]  
(d) \[\frac{-(s^2 + 4)}{(s - 2)^2(s + 2)^2}\]  
(e) \[\frac{3s^2 + 4}{(s^2 - 4)^2}\].

********************
1. The integrating factor for the first order differential equation

\((x^2 - 1) \frac{dy}{dx} = (1 - x)y + x + 1\)

is

(a) \(\frac{1}{x+1}\)  \hspace{1cm} (b) \(x + 1\)  \hspace{1cm} (c) \(\frac{1}{x-1}\)  \hspace{1cm} (d) \(1 - x\)  \hspace{1cm} (e) \(\frac{1}{x^2 - 1}\).

2. Consider the following differential equation

\(P(D)y = e^{3x}x^2\)

where \(P(D) = D^2 + 5D + 4\). The best method for finding the particular solution \(y_p\) is by

(a) \(\frac{e^{3x}x^2}{P(3)}\)  \hspace{1cm} (b) \(x^2 \frac{1}{P(D - 3)} e^{3x}\)

(c) \((4 + (D^2 + 5D) + (D^2 + 5D)^2 + \ldots) e^{3x}x^2\)

(d) \(e^{3x}x^2 \frac{1}{P(-9)}\)  \hspace{1cm} (e) \(e^{3x} \frac{1}{P(D + 3)} x^2\).

3. The value of \(2 \int_0^\infty e^{-st} t \cos^2 2t \, dt\) is

(a) \(\frac{1}{s^2} + \frac{s}{s^2 + 16}\)  \hspace{1cm} (b) \(\frac{4}{s^2 + 16}\)  \hspace{1cm} (c) \(\frac{1}{s} + \frac{s^2 + 16}{(s^2 - 16)^2}\)

(d) \(\frac{1}{s^2} + \frac{s^2 - 16}{(s^2 + 16)^2}\)  \hspace{1cm} (e) \(\frac{2s}{s^2 + 16}\).
4. The value of \( 2 \int_{4}^{\infty} e^{4t-t^2} \) is

(a) \( e^{4\sqrt{\pi}} \operatorname{erfc}(2) \)  
(b) \( e^{-4\sqrt{\pi}} \operatorname{erf}(2) \)  
(c) \( e^{2\sqrt{\pi}} \operatorname{erf}(2) \)  
(d) \( e^{4\sqrt{\pi}} \)  
(e) \( e^{-2\sqrt{\pi}} \operatorname{erfc}(4) \).

5. The second order mixed partial derivative \( \frac{\partial^2 z}{\partial x \partial y} \) of the function \( z = f(x, y) \) given by

\[ z^2 = x^2 e^{2y} \]

is

(a) \( \frac{x^2 e^{2y}}{z^2} \)  
(b) \( \frac{x^2 e^{2y}}{z} \)  
(c) \( \frac{x e^{2y}(2z - 1)}{z^2} \)  
(d) \( \frac{4xe^{2y}(z - 1)}{z^2} \)  
(e) \( \frac{2xe^{2y}}{z} \).

6. Evaluate \( \int_{0}^{2} \int_{y/2}^{1} \cos x^2 \, dx \, dy \).

(a) \( 2 \sin 1 + \frac{\cos 2 - 1}{2} \)  
(b) \( \sin 1 - \cos 1 \)  
(c) \( \sin 2 \)  
(d) \( \sin 1 \)  
(e) \( \frac{\sin^2 1 - \cos^2 1}{2} \).

7. Let \( \mathcal{L}\{f(x)\} = F(s) \), Find \( F(s) \) if

\[ f'(x) - 2 \int_{0}^{x} f(u) \cos 2(x - u) \, du = x \delta(x - 1) \]

where \( f(0) = 1 \).

(a) \( \frac{s^2 - 2}{s} (1 + e^{-s}) \)  
(b) \( \frac{1 - e^{-s}}{s^2 - 4} \)  
(c) \( \frac{s^2 + 4}{s^2 + 2} (1 - e^{-s}) \)  
(d) \( \frac{s^2 + 2}{(s^2 + 4)} (1 + e^{-s}) \)  
(e) \( \frac{s^2 + 4}{s(s^2 + 2)} (1 + e^{-s}) \).

8. The inverse Laplace transform of \( \frac{4e^{-3s}}{s^2 + 4s} \) is

(a) \( e^{4(x-3)} h(x-3) \)  
(b) \( ((x - 3) - e^{3(x-4)}) h(x - 3) \)  
(c) \( (x - 3) h(x - 3) - e^{12-4x} \)  
(d) \( (1 + e^{x-3}) \)  
(e) \( (1 - e^{12-4x}) h(x - 3) \)

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UNIVERSITY OF WOLLONGONG
SCHOOL OF MATHEMATICS AND APPLIED STATISTICS
MATH202 : DIFFERENTIAL EQUATIONS II
PART A: Analytic

Spring Session Examination 1998
Time Allowed: 1 hours 30 minutes

DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. The questions are of equal value.
   Individual parts of the questions may not be of equal value.
3. Notation is as used in lectures.
4. Working (including all necessary reasoning) is to be shown for all solutions.
5. Marks will be deducted for untidy and poorly set out work.

Examination Paper printed on both sides

EXAMINATION MATERIALS/AIDS ALLOWED
Single lined non-alpha numeric calculators are allowed.

EXAMINATION MATERIALS/AIDS TO BE SUPPLIED
None.

THIS EXAMINATION PAPER MUST
NOT BE REMOVED FROM THE
EXAMINATION ROOM
Question 1

(a) Consider the differential equation

\[(\cos x \sin x - xy^2)dx + y(1 - x^2)dy = 0.\]

Determine whether or not the equation is exact. Hence, or otherwise, solve the equation.

(b) Solve the differential equation

\[\frac{dy}{dx} = \frac{2x - y + 2}{2x - y + 3}.\]

(c) Evaluate the following integrals.

(i) \(\int_{0}^{\pi/2} \frac{1}{\sqrt{\tan \theta}} d\theta.\)

(ii) \(\int_{0}^{\infty} z \exp(-z^2) \text{erf}(z) dz.\)

(iii) \(\int_{0}^{\infty} \frac{\cos x}{(x^2 + a^2)} dx,\) by introducing a parameter, and by taking Laplace Transforms with respect to the parameter, where \(a\) is a constant.

(d) Solve the following differential equation subject to the given initial condition,

\[\frac{dy}{dx} - 2xy = 2, \quad y(0) = 1.\]

(e) Solve the integro-differential equation

\[y'(x) + 6y(x) + 9 \int_{0}^{x} y(t) dt = 1, \quad y(0) = 0.\]
Question 2

(a) Prove the result
\[ B(m + 1, n) + B(m, n + 1) = B(m, n). \]

(b) The function \( f \) is defined by \( f(x) = 1 \) for the interval \( 0 < x \leq 1 \).

(i) Find its Fourier sine series.

(ii) On the interval \( -3 < x < 3 \), sketch the function represented by the series in (i).

(iii) On the interval \( -3 < x < 3 \), sketch the function represented by the corresponding Fourier cosine series (do not find this series).

(iv) On the interval \( -3 < x < 3 \), sketch the function represented by the corresponding Fourier series (do not find this series).

(v) Prove that
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)} = \frac{\pi}{4}. \]

(c) Find the eigenvalues and eigenfunctions of
\[ x^2 y'' + xy' + \lambda y = 0. \]
where \( y(1) = 0, \ y(e\pi) = 0 \).

(d) Using the method of separation of variables, solve the boundary value problem
\[
\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0
\]
\[ u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0 \]
\[ u(x, 0) = \sin x + 5 \sin 3x, \quad \frac{\partial u}{\partial t}(x, 0) = 1, \quad 0 < x < \pi. \]

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UNIVERSITY OF WOLLONGONG

SCHOOL OF MATHEMATICS AND APPLIED STATISTICS

MATH202 : DIFFERENTIAL EQUATIONS 2

Spring Session Examination 1999

Time Allowed: 3 hours 15 minutes

DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. The exam paper is broken into two parts: Part A (Analytic) and Part B (Numerical).
   Parts A and B are of equal value.
   PART A: There is one question.
   PART B: There are five questions. The questions are not of equal value.
   Individual parts of the questions may not be of equal value.
3. Equal time should be spent on both PART A and PART B.
4. Use separate books for both PART A and PART B.
5. Examination paper is printed on both sides.
6. Notation is as used in lectures.
7. Working (including all necessary reasoning) is to be shown for all solutions.

EXAMINATION MATERIALS/AIDS ALLOWED

Single lined non-alpha numeric calculators are allowed.

EXAMINATION MATERIALS/AIDS TO BE SUPPLIED

None.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM
Part A  Use separate book

Question 1

(a) (i) Evaluate \( \int_0^\infty u^4e^{-u^2} \, du \).

(ii) Using an integrating factor which is a function of \( y \), or otherwise, solve the differential equation

\[ 2xy \, dx + (4y + 3x^2) \, dy = 0. \]

(iii) Solve the differential equation

\[ x^2 \frac{d^2y}{dx^2} - 2y = \ln x. \]

(b) (i) Solve the integro-differential equation for \( y(t) \) where

\[ y(t) - e^{-t} + 2 \int_0^t y(u) \cosh(t-u) \, du = 0. \]

(ii) Show that \( \mathcal{L}\{xy(x)\} = -\frac{d}{dp}Y(p) \) where \( \mathcal{L}\{y(x)\} = Y(p) \).

Hence, find \( Y(p) \) using the following differential equation.

\[ x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = x^2 \]

where

\( y(0) = 1 \quad \text{and} \quad y'(0) = 0. \)

Note: Find \( Y(p) \) only.

(iii) Given that \( \mathcal{L}\{f(x)\} = F(p) \), show that the Laplace transform of

\[ f(x-a)H(x-a) \quad \text{is} \quad e^{-ap}F(p) \]

where \( H(x) \) is the unit step function.

Hence, or otherwise, find the inverse Laplace transform of

\[ G(p) = \frac{e^{-3p}}{p^2}. \]

(c) (i) Find the Fourier cosine series of the following function

\( f(x) = x, \quad 0 \leq x \leq \pi. \)
(ii) Consider the boundary value problem

\[ \frac{d^2y}{dx^2} + \lambda y = 0 , \quad \lambda > 0 \]

where \( y'(0) = 0 \) and \( y'(\pi) = 0 \).

Find the eigenvalues and corresponding eigenfunctions for the given boundary value problem.

(iii) Using separation of variables and c(i) - c(ii), find a solution in the form of a trigonometric series for the following boundary value problem.

\[ \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} , \quad 0 \leq x \leq \pi , \quad t \geq 0 , \]

where \( z_x(0, t) = z_x(\pi, t) = 0 , \quad t > 0 , \)

and \( z(x, 0) = x , \quad 0 \leq x \leq \pi . \)

END OF PART A
Part B  Use separate book

Question 1

(a) Given the initial value problem

\[ y' - x + y = 0 \quad y(0) = 0. \]

(i) Solve analytically the initial value problem. Find \( y(0.2) \) and \( y(0.4) \).

(ii) Apply the second order Runge-Kutta method to the initial value problem using \( h = 0.2 \) to compute \( y_1, y_2 \).

(iii) Consider the following predictor-corrector method of \( O(h^3) \).

\[
\begin{align*}
    y_{i+1}^* &= y_i + \frac{2}{3} hf(x_i, y_i) \\
    y_{i+1} &= y_i + \frac{h}{4} \left[ f(x_i, y_i) + f \left( x_i + \frac{2}{3} h, y_{i+1}^* \right) \right].
\end{align*}
\]

Perform two iterations using the given initial value problem with a step length of \( h = 0.2 \).

(iv) Compare and discuss your results that you obtain in both (i), (ii) and (iii).

(b) Consider \( y = f(x) \).

(i) Use a Taylor series argument to show that

\[
\begin{align*}
    (\alpha) \quad \frac{dy}{dx} & \text{ is approximated by } \frac{y(x + h) - y(x - h)}{2h} \text{ to } O(h^2) , \\
    (\beta) \quad \frac{d^2y}{dx^2} & \text{ is approximated by } \frac{y(x + h) - 2y(x) + y(x - h)}{h^2} \text{ to } O(h^2). 
\end{align*}
\]

(ii) Consider the following boundary value problem.

\[ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0 \]

subject to \( y(0) = 0 \) and \( y(1) = 1 \).

Use a finite difference approach to calculate \( y = \frac{1}{3} \) and \( y = \frac{2}{3} \) when \([0, 1]\) is divided into 3 equal length intervals.

(iii) Describe the steps that the Linear Shooting method would use to solve

\[ y'' = e^x + 2y + y', \quad y(0) = 2, \quad y(3) = 2. \]

[DO NOT solve the problem.]
(c) Given
\[ \frac{dy}{dx} = f(x, y) \]
subject to \( y(a) = \alpha \).

(i) Derive Euler’s method for solving the given initial value problem.

(ii) State Huen’s method. Discuss the local and global error for this method.

(iii) Using \( f(x, y) = x + y \) and \( y(0) = 1 \), perform two iterations of the Euler and Huen methods where \( h = 0.1 \).

(iv) Briefly explain the advantage of the Runge-Kutta method over the Euler and Huen methods.

(d) Consider the second order initial value problem
\[ y'' = x + y', \quad y(0) = 1, \quad y'(0) = -1. \]

(i) Write the above differential equation as a system of first order differential equations.

(ii) Find \( y(0.1), y(0.2) \) using the Euler’s method using a step length of \( h = 0.1 \).

(e) (i) Find a quadrature formula
\[ \int_{0}^{1} f(x)dx \approx w_{1}f(0) + w_{2}f\left(\frac{1}{3}\right) + w_{3}f(1) \]
and find its order.

(ii) Using part (i) approximate \( \int_{-1}^{1} \sin(\pi x^2)dx. \)

(iii) Estimate the error term for the estimate in part (ii).

(f) (i) Find the rate of convergence as \( n \to \infty \) for the sequence \( \{\left(\sin\frac{1}{n}\right)^2\} \).

(ii) Find the rate of convergence as \( h \to 0 \) for \( \frac{1 - \cos h}{h} \).

(iii) Let \( y = f(x) \).

(\( \alpha \)) Find the first 3 terms of a Taylor series expansion for \( y = f(x) \) about \( x_i \).

(\( \beta \)) Let \( h = x_{i+1} - x_n \). Using (\( \alpha \)) discuss what is meant by global and local error when determining an approximation to \( f(x_{i+1}) \).
(g) Consider the initial value problem \( y' = x^2 - y; \quad y(0) = 1 \) over \([0, 0.2]\).

(i) With \( h = 0.2 \), use the second order Runge-Kutta method to approximate \( y(0.2) \).

(ii) With \( h = 0.1 \), use the second order Runge-Kutta method to approximate \( y(0.1) \) and \( y(0.2) \).

(iii) Given that the exact solution is \( y = x^2 - 2x + 2 - e^{-x} \), calculate the errors in the approximation of \( y(0.2) \) in parts (i) and (ii).

(iv) Does the error behave as expected when \( h \) is halved?

(h) Consider the initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0. \]

(i) Show that the Runge-Kutta scheme

\[ y_{n+1} = y_n + \frac{1}{2} k_1 + \frac{1}{2} k_2 \]

where

\[ k_1 = hf(x_n, y_n) \]
\[ k_2 = hf(x_n + h, y_n + k_1) \]

agrees with the Taylor series expansion

\[ y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} f'(x_n, y_n) \]

up to and including the \( h^2 \) term.

(ii) The solution to the initial value problem

\[ \frac{dy}{dx} = x^3 + y^2, \quad y(0) = 0, \]

has the following values

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0064</td>
</tr>
</tbody>
</table>

Using the fourth order Runge-Kutta method with \( h = 0.2 \), to approximate \( y(0.6) \).
UNIVERSITY OF WOLLONGONG
SCHOOL OF MATHEMATICS AND APPLIED STATISTICS

MATH283 : MATHEMATICS IIE for Engineers Part 1

Differential Equations Part
Autumn Session Examination 2000
Time Allowed: 1 hours 30 minutes
Number of Questions: 2

DIRECTIONS TO CANDIDATES
1. Each question is to be attempted.
2. The questions are of equal value (but individual parts within a question may not be of equal value).
3. Examination paper is printed on both sides.
4. Working (including all necessary reasoning) is to be shown for all solutions.
5. ALL NOTATION is as used in lectures.

EXAMINATION MATERIALS/AIDS ALLOWED
Single line non-alphanumeric keyboard calculators are permitted.

USEFUL INFORMATION
A Tables of Integrals and Laplace Transforms as well as Special Functions are attached.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM
Question 1

(a) (i) Given that \( z^2 + z \cos x + e^{2y^2} = 4 \), find \( \frac{\partial z}{\partial y} \).

(ii) Let \( z = f(u) \) where \( u = \sqrt{x^2 + y^2} \), find \( \frac{\partial z}{\partial x} \) at \( x = 1 \) and \( y = -2 \).

(b) Given \( I = \int_{-1}^{0} \int_{-y}^{y+2} (x + 2y^2) \, dx \, dy \).

(i) Reverse the order of integration for \( I \).

(ii) Evaluate \( I \).

(c) (i) By using a suitable change of variables, evaluate \( \int_{0}^{\infty} u^5 e^{-u^3} \, du \).

(ii) Simplify \( \Gamma \left( -\frac{5}{2} \right) \).

(d) Find the Fourier series of the function defined by

\[ f(x) = x^2, \quad \text{where} \quad 0 \leq x \leq 2\pi. \]

Note: All integrals are to be evaluated.

Question 2

(a) Let \( f(t) = \left( h(t - \frac{\pi}{2}) - h(t) \right) \cos 2t \) where \( h(t) \) is the step function.

(i) Sketch the graph of \( f(t) \).

(ii) Find the Laplace transform of \( f(t) \).

(b) Find the inverse Laplace transform of the following:

(i) \( \frac{1}{s(s + 2)^2} \)

(ii) \( \frac{se^{-3s}}{s^2 + 2s + 5} \).

(c) Solve the integro-differential equation for \( y(t) \) where

\[ y'(x) = \sinh 2x - 4 \int_{0}^{x} y(u) \cosh 2(x - u) \, du \]

subject to \( y(0) = 1 \).

(d) By using Laplace transforms, find the solution of the differential equation

\[ \frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 13y = 4e^{2t} \delta(t - 3) \]

subject to \( y(0) = 0 \) and \( y'(0) = 0 \).

***********************
1. (a) Write down the first three terms of the Taylor series expansion about the point \( x \) for \( y(x + h) \).

(b) Write down the expression for \( y''(x) \) if

\[
y'(x) = \frac{dy}{dx} = f(x, y(x)).
\]

(c) Write down the forward Euler scheme for integrating

\[
\frac{dy}{dx} = f(x, y(x)).
\]

(d) State the local truncation error of the forward Euler scheme.

(e) What answer does the forward Euler scheme give for \( y(h) \) and \( y(2h) \) for the differential equation

\[
\frac{dy}{dx} = x \cdot y, \quad \text{with} \ y(0) = 1.
\]

2. (a) Write down the backward Euler scheme for integrating

\[
\frac{dy}{dx} = f(x, y(x)).
\]

(b) State the local truncation error of this scheme and demonstrate the truth of your answer using the prototype differential equation

\[
\frac{dy}{dx} = \lambda \ y.
\]

(c) Give an example of an explicit ODE integration scheme which has a similar truncation error, and again demonstrate the truth of your answer.

(d) Identify one advantage that the backward Euler scheme has over the explicit scheme that you nominated. Give an example of a differential equation for which this would be important.

(e) Identify one advantage that the explicit scheme that you nominated has over the backward Euler scheme. Give an example of a differential equation for which this would be important.

3. The following is a Runge Kutta integrator with two function evaluations per step

\[
k_1 = h \ f(x, y)
\]

\[
k_2 = h \ f(x + \frac{2}{3} h, \ y + \frac{2}{3} k_1)
\]

\[
y(x + h) = y(x) + \frac{1}{4} k_1 + \frac{3}{4} k_2
\]
(a) For this scheme, identify the local truncation error for the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

by comparing the true solution for \( y(x + h) \) with the Runge Kutta result.

(b) State what you know about the stability properties of this scheme, giving equations where possible. Your working from part (a) should be helpful.

(c) If you believe that stability does impose a limit on the step size \( h \) for this Runge Kutta scheme, what would be that limit for the prototype equation if \( \lambda = -10 \)? If you believe there is no limit, simply write \textbf{no limit}.

4. The trapezoidal scheme for integrating the differential equation

\[
\frac{dy}{dx} = f(x, y(x))
\]

is

\[
y(x + h) = y(x) + \frac{h}{2} \left( f(x, y(x)) + f(x + h, y(x + h)) \right).
\]

(a) Is this scheme implicit or explicit? Explain why.

(b) Apply this scheme to the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

and compare with the Taylor series for \( y(x + h) \) to identify its local truncation error.

(c) State what you know about the stability properties of this scheme, giving equations where possible. Your working from part (b) should be helpful.

(d) If you believe that stability does impose a limit on the step size \( h \) for the trapezoidal scheme, what would be that limit for the prototype equation if \( \lambda = -15 \)? If you believe there is no limit, simply write \textbf{no limit}.

(e) If you had to choose between the trapezoidal method and the backward Euler method, which would you choose, and why?

5. The Adams-Bashforth and Adams-Moulton schemes are both linear multi-step methods. Typical examples of each are

\[
y_{n+1} = y_n + h \left( \frac{1}{2} f_{n+1} + \frac{1}{2} f_n \right)
\]

\[
y_{n+1} = y_n + h \left( \frac{3}{2} f_n - \frac{1}{2} f_{n-1} \right)
\]

(a) In the above notation, what do \( y_n \) and \( f_n \) refer to?

(b) Which of the above two schemes belongs to the Adams-Bashforth family and which to the Adams-Moulton?
(c) What is the local truncation error for each of these two formulae? Use the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

to show that your statements are true.

(d) These two formulae are often combined together to form a predictor-corrector pair. Which will be the predictor, and which the corrector, and why?

(e) Is the predictor-corrector pair easier to use than the trapezoidal rule by itself?

(f) How do you start such a predictor-corrector calculation? You can use the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

to describe what you need to do.

7. The BDF family of linear multi-step methods are commonly used to integrate stiff systems of ordinary differential equations. One family member is

\[
y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2}{3} h f_{n+1}
\]

(a) In the above notation, what do \(y_n\) and \(f_n\) refer to?

(b) Is this BDF family member explicit or implicit? Explain why.

(c) State the local truncation error of this integration scheme, and demonstrate that your assertion is true for the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

(d) Give the value of \(y(2h)\) using the BDF formula above for the differential equation

\[
\frac{dy}{dx} = 3y \quad \text{with} \quad y(0) = 1, \ y(h) = 1 + 3h \quad \text{as starting values}
\]

(e) If you had not been given the value of \(y(h)\) in part (d), how would you start the BDF solution process?
DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. The solution to each question is to be submitted in its own separate, clearly labelled solution book.
3. The two questions are of equal value.
4. Individual parts of the questions may not be of equal value.
5. Equal time should be spent on both questions.
6. Examination paper is printed on both sides.
7. Notation is as used in lectures.
8. Working (including all necessary reasoning) is to be shown for all solutions.

EXAMINATION MATERIALS/AIDS ALLOWED

Single lined non-alphanumeric calculators are allowed.

EXAMINATION MATERIALS/AIDS TO BE SUPPLIED

None.
**Question 1** Use a separate book

(i) Evaluate \( \int_{-2}^{\infty} u e^{-u^2 - 4u} \, du \) in terms of the gamma function.

(ii) Solve the differential equation

\[
(x + 2y - 1) \, dy = (2x - y + 3) \, dx.
\]

(iii) Find \( \mathcal{L}\{\text{erf}(\sqrt{t})\} \).

(ii) Evaluate \( \int_{0}^{\infty} e^{-2t} t \cos 5t \, dt \).

(i) Solve the integral equation for \( g(x) \) where

\[
g(x) = e^{4x} + 2 \int_{0}^{x} \sin 2(x - u)g(u) \, du.
\]

(iii) Let \( \mathcal{L}\{y(x)\} = Y(p) \). Solve the differential equation

\[
y'' + 4y' + 4y = 6\delta(x - 2)
\]

subject to

\[
y(0) = 1 \quad \text{and} \quad y'(0) = 0.
\]

(iii) Using the step function, express the following graph in terms of a periodic function.

\[
\begin{align*}
\text{Graph}
\end{align*}
\]

(ii) Find the Fourier series of the above periodic function.

(iii) Describe the frequency and amplitude characteristics of the different components of the function \( f(t) = \sin 3\pi t - 0.7 \sin 9\pi t \).

(iv) Using the method of separation of variables, solve the following boundary value problem:

\[
\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0
\]

\[
u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0
\]

\[
u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 3 \sin x + 9 \sin 3x, \quad 0 < x < \pi.
\]

*Question 2 starts on the next page...*
Question 2  Use separate book

(a) Consider the following set of data.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.24498</td>
</tr>
<tr>
<td>1.0</td>
<td>0.78540</td>
</tr>
<tr>
<td>1.5</td>
<td>1.15257</td>
</tr>
<tr>
<td>2.0</td>
<td>1.32582</td>
</tr>
</tbody>
</table>

(i) Use an approximation method to evaluate $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ at $x = 1$ correct to $O(h^2)$.

(ii) Using a(i), find a quadratic approximation of $f(x)$ near $x = 1$.

(iii) Determine the rate of convergence for the function $F(h) = \frac{\sin h - h \cos h}{\cos h}$ as $h \to 0$.

(b) Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$ 

(i) Derive Euler’s method for the above initial value problem.

(ii) State Huen’s method along with its local and global errors.

(iii) Let $f(x, y) = 2xy$ with $y(1) = 1$. Using a step length of 0.5 and Huen’s method, determine an approximation to $y(2)$.

(c) Consider the second order initial value problem:

$$(1 - x^2)y'' - 2xy' + 6y = 0, \quad 0 \leq x \leq 1, \quad (*)$$ 

subject to $y(0) = -\frac{1}{2}$ and $y'(0) = 0$.

(i) Write ( * ) as a system of two first order initial value problems.

(ii) Let $h = 0.1$. Use Euler’s method to find approximations to $y(0.1)$ and $y(0.2)$.

(iii) Given that the exact solution is $y = \frac{1}{2}(3x^2 - 1)$, determine the relative percentage error in the approximation of $y(0.1)$. 

continue overpage …
(d) (i) Write down the central finite difference formulas of $O(h^2)$ that approximate the derivatives $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$.

(ii) Consider the second order boundary value problem:

$$\frac{d^2 y}{dx^2} = 16(x^2 - y), \quad 0 \leq x \leq 1,$$

subject to $y(0) = 0$ and $y'(1) = 0$.

(α) Use a finite difference scheme in which $[0, 1]$ is divided into 4 equal length intervals to find approximations to $y(x)$ in $[0, 1]$.

(β) Sketch an approximation to the solution curve for the boundary value problem.

(ii) Describe the steps that the linear shooting method would use to solve

$$y'' = y' + ye^x, \quad y(0) = 2, \quad y(3) = 2.$$

[Do not solve this problem.]
MATH283 - DIFFERENTIAL EQUATIONS

Multiple Choice Test Sample - 1999

Duration - 55 minutes.

Number of questions: 9

All questions of equal value.

Non-alphanumeric calculators with single-line display permitted.

TEST PAPER MUST BE HANDED BACK
WITH COMPUTER SHEET ENCLOSED.

Make sure that the Test Paper and Computer Sheet are filled out correctly.

Failure to do so may lead to zero marks for the Multiple Choice Test.

1. Let \( z = f(u) \), where \( u = \tan(2x + 3y) \). \( \frac{\partial z}{\partial y} \) in terms of \( u \) is

(a) \( f'(u)u^2 \)  \( (b) \ 3f'(u)(1 + u^2) \)  \( (c) \ 2f'(u)(1 + u^2) \)  \( (d) \ 3f'(u) \)  \( (e) \ 3f'(u)(1 - u^2) \).

2. A particular integral for the second order differential equation

\[ y'' + 2y' + y = -4e^{-x}\sin2x \]

is

(a) \( (A \cos x + B \sin x)e^{-x} \)  \( (b) \ (\cos x + \sin x)e^{2x} \)  \( (c) \ e^{-x}\cos2x \)

(d) \( e^{-x}(\cos2x + \sin2x) \)  \( (e) \ e^{-x}\sin2x \).
3. Let $R$ be the region bounded by the curves $y = \sin x$ and $y = \cos x$ for $x$ lying between $x = 0$ and $x = \frac{\pi}{4}$. The value of
\[ \int \int_R ye^x \, dy \, dx \]
is
(a) $\frac{(2 - e^{\pi/4})}{4}$ (b) $(2 - e^{\pi/4})$ (c) $\frac{e^{\pi/4}}{\sqrt{2}} - 1$ (d) $\frac{2e^{\pi/4} - 1}{5}$ (e) $\frac{1}{10}(2e^{\pi/4} - 1)$.

4. Evaluate the total derivative $dz$ when
\[ z = x^2 - 3xy + 2y^2 \] at $x = 2$, $y = -3$, $\Delta x = -0.3$, $\Delta y = 0.2$.

(a) 3.4 (b) 0.3 (c) -2.3 (d) -3.9 (e) -7.5.

5. Find the second order mixed partial derivative $\frac{\partial^2 z}{\partial x \partial y}$ of the function $z = f(x, y)$ given by
\[ z = (x^2 + y^2) \ln(xy) . \]

(a) $\frac{2(x^2 + y^2)}{xy}$ (b) $2x \ln(xy)$ (c) $2x \ln(xy) + \frac{x^2 + y^2}{x}$
(d) $\frac{y^2 + 2xy - x^2}{y^2}$ (e) $x \ln x + y \ln y$.

6. Evaluate $\int_0^\infty x^4 e^{-x^4} \, dx$.

(a) $\frac{\Gamma\left(\frac{1}{4}\right)}{16}$ (b) $\frac{\sqrt{\pi}}{2} \{\text{erf}(4) - \text{erf}(3)\}$ (c) $\frac{\sqrt{\pi}}{2} \text{erf}(4)$ (d) $\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma(5)}$ (e) $\frac{\sqrt{\pi}}{2}$.

7. Evaluate $\int_0^1 e^{-(u+1)^2} \, du$.

(a) $\frac{\Gamma\left(\frac{1}{2}\right)}{2}$ (b) $\frac{\sqrt{\pi}}{2} \{\text{erf}(2) - \text{erf}(1)\}$ (c) $\frac{\sqrt{\pi}}{2} \text{erf}(1)$ (d) $\Gamma\left(\frac{1}{4}\right)$ (e) $\frac{\sqrt{\pi}}{2}$.

8. Let $Y(s) = \mathcal{L}\{y(x)\}$. If
\[ y'(x) + y(x) = h(x-2)e^{-x} \quad \text{where} \quad y(0) = 1 , \]
then the Laplace transform $Y(s)$ is

(a) $\frac{1}{s - 1} + \frac{e^{2s}}{(s + 1)}$ (b) $\frac{e^{-s}}{(s + 1)^2}$ (c) $\frac{-1}{s + 1} + \frac{e^{-(s+2)}}{(s^2 - 1)}$
(d) $\frac{1}{s + 1} + \frac{e^{-2(s+1)}}{(s + 1)^2}$ (e) $\frac{1}{s + 1} + \frac{e^{-2s}}{(s + 1)(s + 2)}$. 

9. Find the Laplace transform of the function \( f(t) = t \cosh 2t \).

\[
\begin{align*}
\text{(a)} & \quad \frac{4 - 2s + s^2}{(s^2 - 4)^2} \\
\text{(b)} & \quad \frac{1}{4 - s^2} \\
\text{(c)} & \quad \frac{s^2 + 4}{(s - 2)^2(s + 2)^2} \\
\text{(d)} & \quad \frac{-(s^2 + 4)}{(s - 2)^2(s + 2)^2} \\
\text{(e)} & \quad \frac{3s^2 + 4}{(s^2 - 4)^2}.
\end{align*}
\]
1. The integrating factor for the first order differential equation

\[(x^2 - 1) \frac{dy}{dx} = (1 - x)y + x + 1\]

is

(a) \(\frac{1}{x + 1}\)  
(b) \(x + 1\)  
(c) \(\frac{1}{x - 1}\)  
(d) \(1 - x\)  
(e) \(\frac{1}{x^2 - 1}\).

2. Consider the following differential equation

\[P(D)y = e^{3x}x^2\]

where \(P(D) = D^2 + 5D + 4\). The best method for finding the particular solution \(y_p\) is by

(a) \(\frac{e^{3x}x^2}{P(3)}\)  
(b) \(x^2 \frac{1}{P(D - 3)} e^{3x}\)  
(c) \((4 + (D^2 + 5D) + (D^2 + 5D)^2 + \ldots) e^{3x} x^2\)  
(d) \(e^{3x} \frac{1}{P(-9)}\)  
(e) \(e^{3x} \frac{1}{P(D + 3)} x^2\).

3. The value of \(2 \int_{0}^{\infty} e^{-2t} t \cos^2 2t \, dt\) is
4. The value of $2 \int_0^\infty e^{4t-t^2}$ is

(a) $e^4 \sqrt{\pi} \operatorname{erfc}(2)$  
(b) $e^{-4} \sqrt{\pi} \operatorname{erf}(2)$  
(c) $e^2 \sqrt{\pi} \operatorname{erf}(2)$  
(d) $e^4 \sqrt{\pi}$  
(e) $e^{-2} \sqrt{\pi} \operatorname{erfc}(4)$.

5. The second order mixed partial derivative $\frac{\partial^2 z}{\partial x \partial y}$ of the function $z = f(x, y)$ given by

$$z^2 = x^2 e^{2y}$$

is

(a) $\frac{x^2 e^{2y}}{z^2}$  
(b) $\frac{x^2 e^{2y}}{z}$  
(c) $\frac{xe^{2y}(2-z)}{z^2}$  
(d) $\frac{4xe^{2y}(z-1)}{z^2}$  
(e) $\frac{2xe^{2y}}{z}$.

6. Evaluate $\int_0^2 \int_{y/2}^1 \cos x^2 \, dx \, dy$.

(a) $2 \sin 1 + \frac{\cos 2 - 1}{2}$  
(b) $\sin 1 - \cos 1$  
(c) $\sin 2$  
(d) $\sin 1$  
(e) $\sin^2 1 - \cos^2 1$.

7. Let $\mathcal{L}\{f(x)\} = F(s)$. Find $F(s)$ if

$$f'(x) - 2 \int_0^x f(u) \cos 2(x-u) \, du = x\delta(x-1)$$

where $f(0) = 1$.

(a) $\frac{s^2 - 2}{s} (1 + e^{-s})$  
(b) $\frac{1 - e^{-s}}{s^2 - 4}$  
(c) $\frac{s^2 + 4}{s^2 + 2} (1 - e^{-s})$  
(d) $\frac{s^2 + 2}{s^2 + 4} (1 + e^{-s})$  
(e) $\frac{s^2 + 4}{s^2 + 2} (1 + e^{-s})$. 
8. The inverse Laplace transform of \( \frac{4e^{-3s}}{s^2 + 4s} \) is

(a) \( e^{4(x-3)}h(x-3) \)  
(b) \( (x - 3) - e^{3(x-4)}h(x - 3) \)  
(c) \( (x - 3)h(x - 3) - e^{12-4x} \)

(d) \( 1 + e^{x-3} \)  
(e) \( 1 - e^{12-4x}h(x - 3) \)

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UNIVERSITY OF WOLLONGONG
SCHOOL OF MATHEMATICS AND APPLIED STATISTICS

MATH283 : MATHEMATICS IIE for Engineers Part 1

Autumn Session Examination 1999
Time Allowed: 2 hours 15 minutes
Number of Questions: 2

DIRECTIONS TO CANDIDATES
1. Each question is to be attempted.
2. The questions are of equal value (but individual parts within a question may not be of equal value).
3. Examination paper is printed on both sides.
4. Working (including all necessary reasoning) is to be shown for all solutions.
5. ALL NOTATION is as used in lectures.

EXAMINATION MATERIALS/AIDS ALLOWED
Non-alphanumeric calculators with single line display are permitted.

USEFUL INFORMATION
A Table of Integrals and a Table of Laplace Transforms are attached.

THIS EXAMINATION PAPER MUST NOT BE REMOVED
FROM THE
EXAMINATION ROOM
Question 1

(a) Find the Fourier sine series for the function \( f(x) = x \), \( 0 \leq x \leq \pi \).

(b) Consider the boundary value problem
\[
\frac{d^2 y}{dx^2} + \lambda y = 0, \quad \lambda > 0
\]
where \( y(0) = 0 \) and \( y(\pi) = 0 \).

Find the eigenvalues and corresponding eigenfunctions for the given boundary value problem.

(c) Using separation of variables, find the trigonometric series for the following boundary value problem.
\[
\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \quad 0 \leq x \leq \pi, \quad t \geq 0,
\]
where \( z(0, t) = z(\pi, t) = 0, \quad t > 0 \), and \( z(x, 0) = x, \quad 0 \leq x \leq \pi \).

Question 2

(a) Solve the initial value problem
\[
\frac{dy}{dx} = \frac{1}{1 + y}, \quad y(0) = 0.
\]

(b) Let \( X(s) = \mathcal{L}\{x(t)\} \) and \( Y(s) = \mathcal{L}\{y(t)\} \). Using the following system of differential equations, find \( X(s) \).
\[
\frac{d^2 x}{dt^2} + \frac{dy}{dt} = 0,
\]
\[
\frac{d^2 y}{dt^2} - \frac{dx}{dt} = h(t)e^{-2t},
\]
where \( h(t) \) is the step function and \( x'(0) = y'(0) = x(0) = y(0) = 0 \).

(c) Find the inverse Laplace transform of
\[
G(s) = \frac{2s}{(s^2 + 1)(s^2 + 4)} + \frac{e^{-2s}}{s^2}.
\]

(d) Solve the integral equation for \( y(t) \) where
\[
y(t) - e^{-t} + 2 \int_0^t y(u) \sinh(t - u) \, du = 0.
\]
Differential Equations Part

Autumn Session Examination 2000

Time Allowed: 1 hours 30 minutes
Number of Questions: 2

DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. The questions are of equal value (but individual parts within a question may not be of equal value).
3. Examination paper is printed on both sides.
4. Working (including all necessary reasoning) is to be shown for all solutions.
5. ALL NOTATION is as used in lectures.

EXAMINATION MATERIALS/AIDS ALLOWED

Single line non-alphanumeric keyboard calculators are permitted.

USEFUL INFORMATION

A Tables of Integrals and Laplace Transforms as well as Special Functions are attached.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM
Question 1
(a) (i) Given that \( z^2 + z \cos x + e^{xy^2} = 4 \), find \( \frac{\partial z}{\partial y} \).
(ii) Let \( z = f(u) \) where \( u = \sqrt{x^2 + y^2} \), find \( \frac{\partial z}{\partial x} \) at \( x = 1 \) and \( y = -2 \).

(b) Given \( I = \int_{-1}^{0} \int_{-y}^{y+2} (x + 2y^2) \, dx \, dy \).
(i) Reverse the order of integration for \( I \).
(ii) Evaluate \( I \).

(c) (i) By using a suitable change of variables, evaluate \( \int_0^{\infty} u^5 e^{-u^3} \, du \).
(ii) Simplify \( \Gamma \left( -\frac{5}{2} \right) \).

(d) Find the Fourier series of the function defined by
\[
f(x) = \sin x, \quad 0 \leq x \leq 2\pi.
\]

Note: All integrals are to be evaluated.

Question 2
(a) Let \( f(t) = \left( h(t - \frac{\pi}{2}) - h(t) \right) \cos 2t \) where \( h(t) \) is the step function.
(i) Sketch the graph of \( f(t) \).
(ii) Find the Laplace transform of \( f(t) \).

(b) Find the inverse Laplace transform of the following:
(i) \( \frac{1}{s(s + 2)^2} \)
(ii) \( \frac{se^{-3s}}{s^2 + 2s + 5} \).

(c) Solve the integro-differential equation for \( y(t) \) where
\[
y'(x) = \sinh 2x - 4 \int_0^x y(u) \cosh 2(x - u) \, du
\]
subject to \( y(0) = 1 \).

(d) By using Laplace transforms, find the solution of the differential equation
\[
\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 13y = 4e^{2t}\delta(t - 3)
\]
subject to \( y(0) = 0 \) and \( y'(0) = 0 \).

***********************
1. (a) Write down the first three terms of the Taylor series expansion about the point \( x \) for \( y(x + h) \).

(b) Write down the expression for \( y''(x) \) if

\[
y'(x) = \frac{dy}{dx} = f(x, y(x)).
\]

(c) Write down the forward Euler scheme for integrating

\[
\frac{dy}{dx} = f(x, y(x))
\]

(d) State the local truncation error of the forward Euler scheme.

(e) What answer does the forward Euler scheme give for \( y(h) \) and \( y(2h) \) for the differential equation

\[
\frac{dy}{dx} = x \cdot y, \text{ with } y(0) = 1
\]

2. (a) Write down the backward Euler scheme for integrating

\[
\frac{dy}{dx} = f(x, y(x))
\]

(b) State the local truncation error of this scheme and demonstrate the truth of your answer using the prototype differential equation

\[
\frac{dy}{dx} = \lambda \cdot y
\]

(c) Give an example of an explicit ODE integration scheme which has a similar truncation error, and again demonstrate the truth of your answer.

(d) Identify one advantage that the backward Euler scheme has over the explicit scheme that you nominated. Give an example of a differential equation for which this would be important.

(e) Identify one advantage that the explicit scheme that you nominated has over the backward Euler scheme. Give an example of a differential equation for which this would be important.

3. The following is a Runge Kutta integrator with two function evaluations per step

\[
k_1 = h \cdot f(x, y)
k_2 = h \cdot f(x + \frac{2}{3}h, y + \frac{2}{3}k_1)
\]

\[
y(x + h) = y(x) + \frac{1}{4}k_1 + \frac{3}{4}k_2
\]

(a) For this scheme, identify the local truncation error for the prototype equation

\[
\frac{dy}{dx} = \lambda \cdot y \quad \text{with} \quad y(0) = 1
\]

by comparing the true solution for \( y(x + h) \) with the Runge Kutta result.

(b) State what you know about the stability properties of this scheme, giving equations where possible. Your working from part (a) should be helpful.
(c) If you believe that stability does impose a limit on the step size \( h \) for this Runge Kutta scheme, what would be that limit for the prototype equation if \( \lambda = -10 \)? If you believe there is no limit, simply write no limit.

4. The trapezoidal scheme for integrating the differential equation

\[
\frac{dy}{dx} = f(x, y(x))
\]

is

\[
y(x + h) = y(x) + \frac{h}{2} \left( f(x, y(x)) + f(x + h, y(x + h)) \right)
\]

(a) Is this scheme implicit or explicit? Explain why.

(b) Apply this scheme to the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

and compare with the Taylor series for \( y(x+h) \) to identify its local truncation error.

(c) State what you know about the stability properties of this scheme, giving equations where possible. Your working from part (b) should be helpful.

(d) If you believe that stability does impose a limit on the step size \( h \) for the trapezoidal scheme, what would be that limit for the prototype equation if \( \lambda = -15 \)? If you believe there is no limit, simply write no limit.

(e) If you had to choose between the trapezoidal method and the backward Euler method, which would you choose, and why?

5. The Adams-Bashforth and Adams-Moulton schemes are both linear multi-step methods. Typical examples of each are

\[
y_{n+1} = y_n + h \left( \frac{1}{2} f_{n+1} + \frac{1}{2} f_n \right)
\]

\[
y_{n+1} = y_n + h \left( \frac{3}{2} f_n - \frac{1}{2} f_{n-1} \right)
\]

(a) In the above notation, what do \( y_n \) and \( f_n \) refer to?

(b) Which of the above two schemes belongs to the Adams-Bashforth family and which to the Adams-Moulton?

(c) What is the local truncation error for each of these two formulae? Use the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

to show that your statements are true.

(d) These two formulae are often combined together to form a predictor-corrector pair. Which will be the predictor, and which the corrector, and why?

(e) Is the predictor-corrector pair easier to use than the trapezoidal rule by itself?

(f) How do you start such a predictor-corrector calculation? You can use the prototype equation

\[
\frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1
\]

to describe what you need to do.
6. The BDF family of linear multi-step methods are commonly used to integrate stiff systems of ordinary differential equations. One family member is

\[ y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2}{3} h f_{n+1} \]

(a) In the above notation, what do \( y_n \) and \( f_n \) refer to?

(b) Is this BDF family member explicit or implicit? Explain why.

(c) State the local truncation error of this integration scheme, and demonstrate that your assertion is true for the prototype equation

\[ \frac{dy}{dx} = \lambda y \quad \text{with} \quad y(0) = 1 \]

(d) Give the value of \( y(2h) \) using the BDF formula above for the differential equation

\[ \frac{dy}{dx} = 3y \quad \text{with} \quad y(0) = 1, \quad y(h) = 1 + 3h \] as starting values

(e) If you had not been given the value of \( y(h) \) in part (d), how would you start the BDF solution process?
Chapter 11: References


Chapter 12: Solutions

Exercise 1A

1 (a) cot \( x \)

(b) \( 2x(1 + y^2)e^{x^2} \)

(c) \( -\frac{2x}{(x^2 + 1)^2} f'(v) \)

(d) \( \frac{1 - 2x(x + y)}{2y(x + y) - 1} \)

2 (a) \( \frac{\partial z}{\partial x} = 12x^2y \)

\( \frac{\partial z}{\partial y} = 4x^3 \)

(b) \( \frac{\partial z}{\partial x} = 2x + y \)

\( \frac{\partial z}{\partial y} = -9y^2 + x \)

(c) \( \frac{\partial z}{\partial x} = xy^2e^{x^2}y^2 \)

\( \frac{\partial z}{\partial y} = x^3ye^{x^2}y^2 \)

(d) \( \frac{\partial z}{\partial x} = 3x^2y^2 \cos (x^3y^2) \)

\( \frac{\partial z}{\partial y} = 2x^3y \cos (x^3y^2) \)

(e) \( \frac{\partial z}{\partial x} = e^{xy}(y \cos x^2 - 2x \sin x^2) \)

\( \frac{\partial z}{\partial y} = xe^{xy} \cos x^2 \)

Exercise 1B

1 (a) \( \frac{\partial z}{\partial x} = 12x^2y \)

\( \frac{\partial z}{\partial y} = 4x^3 \)

\( \frac{\partial^2 z}{\partial x^2} = 24xy \)

\( \frac{\partial^2 z}{\partial y^2} = 0 \)

\( \frac{\partial^2 z}{\partial x \partial y} = 12x^2 \)

(b) \( z_x = 2f'(5) + 3g'(1) \)

\( z_y = 3f'(5) - 4g'(1) \)

\( z_{xx} = 4f''(5) + 9g''(1) \)

\( z_{yy} = 9f''(5) + 16g''(1) \)

\( z_{xy} = 6f''(5) - 12g''(1) \)

(d) \( z_u = \frac{2}{2u + v} \)

\( z_v = \frac{1}{2u + v} \)

\( z_{uu} = -\frac{4}{(2u + v)^2} \)

\( z_{uv} = -\frac{1}{(2u + v)^2} \)

\( z_{vv} = -\frac{2}{(2u + v)^2} \)

continued next page...
4 (b) $\rho_\phi = \cos \phi \cos \theta$
\[ \rho \theta = -\sin \phi \sin \theta \]
\[ \rho_{\phi\phi} = -\sin \phi \cos \theta \]
\[ \rho_{\theta \theta} = -\sin \phi \cos \theta \]
\[ \rho_{\theta \phi} = -\cos \phi \sin \theta \]

**Exercise 1C**

1 $\Delta z = -7.15 \quad dz = -7.5$

2 (a) $dz = ydx + xdy$
(b) $dz = \left( \frac{y}{x} dx + \ln x dy \right) z$
(c) $dz = \frac{1}{x} dx + \frac{1}{y} dy$
(d) $dz = -\sin x \sin y dx + \cos x \cos y dy$
(e) $dz = y \sec^2 x + \tan x dy$

3 Let $z = f(x, y) = x^\frac{1}{2} y^\frac{1}{2}$ where $x = 25, y = 1000$ and $dx = 2, dy = 21.$

7 (a) $f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + c$
(b) $f(x, y) = x^2 y + c$
(c) $f(x, y) = xe^y + c$

**Exercise 1D**

1 (a) 1
(b) $\frac{1}{s^2 + 1}$
(c) $\sin^{-1} t$  (d) $\sqrt{5} - 1$
(e) $\tan x + c$  (f) 2
(g) $u \cos^{-1} u - \sqrt{1 - u^2}$
(h) $e^t - t - 1$
(i) $(u - t) \cos u - \sin u|_0^t = \sin t - t$

2 (a) $\frac{1}{12}$  (b) $\frac{1}{20}$
(c) $\frac{4}{3}$

3 (a) $\frac{1}{2e}(c - 1)$  (b) 4

(c) 2

4 (a) $\frac{1}{6}$

5 (a) $\frac{1}{24}$

**Exercise 2B**

1 $x^2(1 - y^2) = ky^2(1 - x^2)$

2 $1 - y^2 = k(1 - x^2)$

3 $y + 1 = k \epsilon^x (x - 1)$

4 $yx^x = k \epsilon^{x - 2y}$

5 $e^{2y} = 2e^x + k$

**Exercise 2C**

1 $y + \sqrt{x^2 + y^2} = kx^2$

2 $x + 2(x + 2y) \ln kx = 0$

3 $x^2 + y^2 = cx$

4 $x^2 + 2xy = c$

5 $\tan^{-1} \frac{y + 3}{x + 2} + \frac{1}{2} \ln(x^2 + y^2 4x + 6y + 13) = c$

6 $x^2 + y^2 - xy - 4x + 5y + c = 0$

7 $(y - x - 1)^2 + 2x - 1 = c$

**Exercise 2D**

1 $y = -\cos x + c \sec x$ or $y = \sin x \tan x + k \sec x$

2 $y \sin x = -\frac{1}{8} \cos 4x + \frac{1}{4} \cos 2x + c$ or $y \sin x = -\cos 4x - \frac{3}{2} \sin 2x + k$

continued next page...
Exercise 2E
(a) \( y = x^3 \log x \)
(b) \( \frac{x}{1+x}(2x + x^2 + c_1) \)
(c) \( \frac{1}{x(e^x + c)} \)

Exercise 2F
1 (a) \( x^2 + 3x + y^2 - 2y = c \)

3 (a) \( y^3 - xy = c \)
(b) \( e^{xy} - x^2y = c \)
(c) \( ye^x - x^3 = c \)
(d) \( \sin x = k \cos y \)

Exercise 2G
2 (a) \( R(x) = e^{3x}, \quad (3x^2y + y^3)e^{3x} = c \)
(b) \( R(y) = y, \quad xy + y \cos y - \sin y = c \)
(c) \( R(x, y) = xy, \quad x^3y + 3x^2y^3 = c \)

Exercise 2H
1 (a) \( y = 3 + c(1 - x^2)^{1/2} \)
(b) \( y + \sqrt{(y^2 - x^2)} = cx^3 \)
(c) \( y^2 \ln y - x = cy^2 \)
(d) \( y = \frac{1}{\sin x + c \cos x} \)
(e) \( \frac{1}{y} = \frac{1}{x} - \frac{3}{2x^2} + c \)
(f) \( x^2 + y^2 - e^{xy} = c \)

(g) \( y + \frac{1}{y} + (x - 2)e^x = c \)
(h) \( y = x \ln(1 + x) + cx \)
(i) \( yx^3 = \frac{1}{2}x^2 + c \)
(j) \( y^{1/2} = -2 - x + ce^{x/2} \)
(k) \( x^2y + xe^y = c \)

3 (a) \( \ln(2x^2 - y^2 - 4x - 4y - 2) = \frac{3}{\sqrt{2}} \ln \left( \frac{\sqrt{2}x + y + 2 - \sqrt{2}}{\sqrt{2}x - y - 2 + \sqrt{2}} \right) \)
(b) \( y = \frac{cx}{(1 + x)\sqrt{1 - x^2}} \)
(c) \( xy^2 = 1 + ce^{-x} \)
(d) \( yx^3 = \frac{1}{2}x^2 + c \)

Exercise 3A
1 (a) \( y + \frac{1}{6}x^2 + cx + d = 0 \)
(b) \( y = \ln \sec x + cx + d \)
(c) \( \pm c \ln(y + \sqrt{y^2 + c^2}) = x + d \)
(d) \( y = c \quad \text{or} \quad y = a \tan(\frac{1}{2}ax + b) \)
(e) \( y = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + cx + d \)
(f) \( y = c \quad \text{or} \quad y = -\ln x + c \)
(h) \( y = a \cos nx + b \sin nx \)

continued next page...
2 (a) \( y = (ax + b)e^{-4x} \)
(b) \( y = a\sqrt{1 + x^2} + cx \)
(c) \( y = ae^{-x} + be^{5x} \)
(d) \( y = a\cos 2x + b\sin 2x \)
(e) \( y = ae^x + be^{3x} + x \)

Exercise 3B

1 \( y = Ae^{4x} + Be^{-2x} - \frac{1}{2} \)
2 \( y = Ae^{-x/2} + Be^{-2x} + \frac{1}{2}x^2 - \frac{5}{2}x + 5\frac{1}{4} \)
3 \( y = (A + Bx)e^{-4x} + e^x \)
4 \( y = A + Be^{-4x} + x^2 - \frac{1}{2}x \)
\[ -\frac{3}{10}(2\cos 2x + \sin 2x) \]
5 \( y = \frac{1}{2}e^{-x} + (A + Bx)e^{-3x} \)
6 \( y = \frac{1}{4}x\sin 2x + A\cos 2x + B\sin 2x \)
7 \( y = \sin x + Ae^{x/2} + Be^{-x} \)
8 \( y = \frac{1}{2}x\sinh x + Ae^x + Be^{-x} \)
9 \( y = \frac{1}{6}x^3e^x + (A + Bx)e^x + \frac{3e^{-x}}{2} + e^{-3x} \)
10 \( y = \frac{8}{3}x^3e^{-2x} + (A + Bx)e^{-2x} \)
11 \( y = \frac{2}{3}x^2e^x + \frac{5}{9}xe^x + Ae^x + Be^{-2x} \)
12 \( y = x^4e^{-2x} + (A + Bx)e^{-2x} \)
13 \( y = (Ax + B + 2x^2)e^{2x} \)

Exercise 4A

1 (a) \( a^3e^{ax}(a^3x^3 + 18a^2x^2 + 90ax + 120) \)
\( 16x^2\cos 2x + 64x\sin 2x - 48\cos 2x \)
\( e^{2x}(120\sin 3x - 119\cos 3x) \)
(c) (i) \( \frac{(2n)!}{n!} \) ; \( 0 \)
(ii) \( (-1)^n ; (-1)^{n-1}n^2 \)
3 \( I(a) = \frac{\sqrt{\pi}e^{-a^2/4}}{2} \)
4 \( \frac{(-1)^nn!}{(x+1)^{n+1}} \)

Exercise 4B

1 (a) \( 4 \) ; (b) \( \frac{15}{4} \) (c) \(-2\sqrt{\pi} \)
(d) \(-8\sqrt{\pi}/15 \)
2 (a) \( 24 \) (b) \( \frac{3}{4} \) (c) \( \sqrt{\pi}/4 \)
(d) \( \frac{\Gamma\left(\frac{1}{4}\right)}{3} \)
(f) \( 5e^{-1} \)

Exercise 4C

1 (a) \( \frac{\Gamma\left(\frac{7}{5}\right)}{5} \) (b) \( \frac{2}{33} \) (c) \( \frac{1}{40} \)
(d) \( \frac{1}{2}\Gamma\left(\frac{3}{4}\right)^2\Gamma\left(\frac{1}{4}\right)^3 \) (e) \( \frac{1}{252} \)
(f) \( \Gamma\left(\frac{5}{3}\right) \) (g) \( \sqrt{\frac{4\pi}{5}} \) (h) \( \sqrt{\pi} \)
continued next page...
(i) 29  \hspace{1cm} (j) \sqrt{\pi} \frac{\Gamma\left(\frac{1}{4}\right)}{3\Gamma\left(\frac{3}{4}\right)} \hspace{1cm} (k) \frac{3\pi}{512}

(l) \frac{1}{4}\sqrt{\pi} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - 4 \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)

(m) \frac{\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)} \hspace{1cm} (n) \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{4\sqrt{\pi}}

(p) \frac{e^4}{2}(1 - 2\sqrt{\pi})

Exercise 4D

2 (a) \frac{1}{\sqrt{\pi}}

4 (a) \frac{e^{-2s}}{s} \hspace{1cm} (b) \frac{1}{s}

(d) \frac{1}{s}(1 + e^{-2s}) \hspace{1cm} (e) \sin 3e^{-s}

Exercise 5A

1 (b) (iii) \frac{n}{(p-1)^2 + n^2}

(iv) \frac{p-1}{(p-1)^2 + n^2}

(vi) \frac{p-3+n}{(p-3)^2 + n^2}

(vii) \frac{6p^2}{(p^2-9)^2} + \frac{p^2 - 4}{(p^2 + 4)^2}

Exercise 5B

2 (a) \frac{e^{-2p}}{p}

(b) \frac{e^{-2p}}{p - 1}

(c) \frac{e^{2(1-p)}}{p - 1}

(d) \frac{-1}{p^2 + 9} \left(3 + pe^{-\frac{2p}{3}}\right)

(e) \frac{e^{-5p}}{p}

(f) \frac{\sqrt{p + 1} - 1}{p\sqrt{p + 1}}

(g) \frac{3}{p(p^2 + 9)}

continued next page...
(h) \( \frac{1}{p^2 + 4} \)

(i) \( \frac{\sqrt{\pi} e^{\frac{p^2}{2}}}{2} \text{erf} \left( \frac{p}{2} \right) \)

(j) \( \frac{3}{p^2 + 9} \)

(k) \( \frac{p}{p^4 + 4} \)

(l) \( \frac{3e^{-2p}}{p(p^4 + 9)} \)

(m) \( \frac{e^{-3p}}{p^2 + 4} \)

3 \( y(x) = y(0)e^{-x} + e^{-(x-1)}h(x-1) \)

Exercise 5C

1 (a) \( \frac{2}{3} e^{-2t/3} \)

(b) \( \frac{5}{6} e^{t/2} - \frac{1}{3} e^{-t} \)

(c) \( \frac{1}{4} - \frac{1}{4} \cos 2t \)

(d) \( \left( \frac{9}{2} t^2 - 6t + 1 \right) e^{-3t} \)

(e) \( \frac{1}{4} t - \frac{1}{3} \sin t + \frac{1}{24} \sin 2t \)

(f) \( -e^{-3t}(\cos 2t + \sin 2t) - t + 1 \)

2 (b) (i) \( (x-a)h(x-a) \)

(ii) \( \sin(x - \pi)h(x - \pi) \)

(iii) \( \frac{1}{2} e^{-(t-4)} \sin 2(t-4) h(t-4) \)

Exercise 5D

1 (a) \( x = 1 + e^{-3t} - 2e^{-4t} \)

(b) \( x = -e^{-t} + 2e^{-2t} - 2te^{-2t} \)

(c) \( x = e^{-t}(\cos t + \sin t) \)

(d) \( x = \frac{1}{2} \cos t - \frac{1}{2} \sin t + \frac{3}{2} e^t \)

2 (a) \( \frac{1}{9}(1 - \cos 3t) \)

(b) \( \frac{2}{3}(\cos t - \cos 2t) \)

(c) \( \frac{\sin 2t}{4} - \frac{t}{2} \)

(d) \( \frac{1}{2n^2} \left[ \frac{\sin nt}{n} - t \cos nt \right] \)

4 (b) (i) \( \frac{\pi}{2} \)

(ii) \( \ln \frac{b}{a} \)

5 (a) \( g(x) = x \)

(b) \( g(x) = (1 + x)^2 e^x \)

6 (a) \( y = (y(0) - 1) \cosh x + 1 \)

(b) \( f(x) = f(0) \left( 1 + \frac{1}{2} k^2 x^2 \right) \)

Exercise 5E

1 (a) \( x = e^{-2t} \cos 3t \); \( y = e^{-2t} \sin 3t \)

(b) \( x = 2 \cos t + 2 \sin t \); \( y = 4 \sin t \)

2 (a) \( x = \frac{1}{3}(e^t - e^{-2t}) \); \( y = \frac{1}{3}(e^{-2t} + e^t) \)

(b) \( x = \frac{t}{8} - \frac{e^t}{15} + \frac{173e^{4t}}{192} + \frac{53e^{-4t}}{320} \)

\( y = \frac{1}{16} - \frac{8e^t}{15} + \frac{173e^{4t}}{96} - \frac{53e^{-4t}}{160} \)

continued next page...
Exercise 6B

1. (a) (i) \[ \frac{1}{2} + \frac{2}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots \right) \]
    
(b) (ii) \[ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2}} \]
    
(c) (ii) \[ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2}} - \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n} \]
    
(d) (iv) \[ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2}} - \sum_{n=1}^{\infty} \frac{\sin nx}{n} \]
    
(e) (iv) \[ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2}} - \sum_{n=1}^{\infty} \frac{\sin nx}{n} \]

Exercise 7A

(a) \( \lambda_n = n^2 \), \( y_n = \sin nx \);
    \[ n = \pm 1, \pm 2, \pm 3, \pm 4 \ldots \]

(b) \( \lambda_n = n^2 \), \( y_n = \cos nx \);
    \[ n = 0, 1, 2, \ldots \]

(c) \( \lambda_n = \pm \left( \frac{2n+1}{2} \pi \right)^2 \),
    \[ y_n = \sin \left( \frac{(2n+1)\pi x}{2} \right) \quad ; \quad n = 0, 1, 2, \ldots \]

(d) \( \lambda_0 = 0 \), \( y_0 = 1 - x \)
    \[ \lambda_n = -\mu_n^2 \), \( y_n = \sin \mu_n x - \mu_n \cos \mu_n x \); \[ n = 1, 2, 3, \ldots \]

where \( \mu_n \) are the roots of \( \mu_n = \tan \mu_n \).

(e) \( \lambda_n = \mu_n^2 \), \( y_n = \sin \mu_n x \);
    \[ n = 0, 1, 2, 3 \ldots \]

where \( \mu_n \) are the roots of
    \[ \mu_n = -\tan \mu_n \pi. \]

Exercise 7B

1. \( U = 5e^{-16x^2} \sin 2\pi x \)

2. \( U = 2e^{-4.5x} \sin 3x - 5e^{-8x} \sin 4x \)

3. \( Y = 10 \sin \frac{\pi x}{2} \cos \frac{\pi x}{2} \)

4. \( Y = 6 \sin x \sin \frac{t}{3} - 4.5 \sin 2x \sin \frac{2t}{3} \)
Exercise 7C

1(a) \( u(x, t) = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 \pi^2 t} \sin(2n-1) \pi x}{(2n-1)^3} \)

(b) \( u(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 \pi^2 t} \sin(2n-1) \pi x}{(2n-1)} \)

(c) \( u(x, t) = \frac{-4}{\pi^2} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 \pi^2 t} \cos(2n-1) \pi x}{(2n-1)^2} \)

2 \( u(x, y) = \frac{4a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{(1 - \cos n \pi)}{n^3} \frac{e^{-n^2 \pi y/a}}{e^{-n \pi u \sin (n \pi x/a)}} \)
# Table of Integrals

1. \[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1 \]
2. \[ \int \frac{dx}{x} = \ln |x| + c \]
3. \[ \int e^x \, dx = e^x + c \]
4. \[ \int \sin x \, dx = -\cos x + c \]
5. \[ \int \cos x \, dx = \sin x + c \]
6. \[ \int \tan x \, dx = \ln |\sec x| + c \]
7. \[ \int \sec^2 x \, dx = \tan x + c \]
8. \[ \int \csc^2 x \, dx = -\cot x + c \]
9. \[ \int \sinh x \, dx = \cosh x + c \]
10. \[ \int \cosh x \, dx = \sinh x + c \]
11. \[ \int \tanh x \, dx = \ln(\cosh x) + c \]
12. \[ \int (ax + b)^n \, dx = \frac{1}{a^2} (ax + b)^{n+1} \left[ \frac{ax + b}{n+2} - \frac{b}{n+1} \right] + c, \quad n \neq -1, -2 \]
13. \[ \int \frac{x^2}{ax + b} \, dx = \frac{1}{a^3} \left[ \frac{1}{2} (ax + b)^2 - 2b(ax + b) + b^2 \ln |ax + b| \right] + c \]
14. \[ \int \frac{x^2}{(ax + b)^2} \, dx = \frac{1}{a^2} \left[ ax + b - \frac{b^2}{ax + b} - 2b \ln |ax + b| \right] + c \]
15. \[ \int \frac{x}{\sqrt{ax + b}} \, dx = \frac{2}{a^2} \left[ \frac{(ax + b)^{5/2}}{5} - \frac{b(ax + b)^{3/2}}{3} \right] + c \]
16. \[ \int \frac{1}{x \sqrt{ax + b}} \, dx = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax + b} - \sqrt{b}}{\sqrt{ax + b} + \sqrt{b}} \right| + c, \quad b > 0 \]
17. \[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + c \]
18. \[ \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + c \]
19. \[ \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln \left| \frac{x + a}{x - a} \right| + c \]
\[ \int \frac{1}{(a^2 - x^2)^2} \, dx = \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{4a^3} \ln \left| \frac{x + a}{x - a} \right| + c \]
\[ \int \frac{1}{x \sqrt{a^2 - x^2}} \, dx = -\frac{1}{a} \ln \left| a + \sqrt{a^2 - x^2} \right| + c \]
\[ \int \frac{1}{(a^2 - x^2)^{3/2}} \, dx = \frac{x}{a^2 \sqrt{a^2 - x^2}} + c \]
\[ \int \frac{\sqrt{a^2 - x^2}}{x} \, dx = \sqrt{a^2 - x^2} - a \ln \left| a + \sqrt{a^2 - x^2} \right| + c \]
\[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left| x + \sqrt{x^2 + a^2} \right| + c \]
\[ \int \frac{1}{x \sqrt{x^2 + a^2}} \, dx = -\frac{1}{a} \ln \left| a + \sqrt{x^2 + a^2} \right| + c \]
\[ \int \frac{1}{(x^2 + a^2)^{3/2}} \, dx = \pm \frac{x}{a^2 \sqrt{x^2 + a^2}} + c \]
\[ \int \sqrt{x^2 \pm a^2} \, dx = \frac{1}{2} \sqrt{x^2 \pm a^2} \pm \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| + c \]
\[ \int \frac{\sqrt{x^2 + a^2}}{x} \, dx = \sqrt{x^2 + a^2} - a \ln \left| a + \sqrt{x^2 + a^2} \right| + c \]
\[ \int \frac{1}{b + ke^{ax}} \, dx = \frac{1}{ab} \left[ ax - \ln(b + ke^{ax}) \right] + c, \quad ab \neq 0 \]
\[ \int e^{ax} \sin bx \, dx = \frac{1}{a^2 + b^2} e^{ax} (a \sin bx - b \cos bx) + c \]
\[ \int e^{ax} \cos bx \, dx = \frac{1}{a^2 + b^2} e^{ax} (a \cos bx + b \sin bx) + c \]
\[ \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n - 1}{n} \int \sin^{n-2} x \, dx \]
\[ \int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n - 1}{n} \int \cos^{n-2} x \, dx \]
\[ \int \tan^n x \, dx = \frac{1}{n - 1} \tan^{n-1} x - \int \tan^{n-2} x \, dx \]
\[ \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n - 1} + \frac{n - 2}{n - 1} \int \sec^{n-2} x \, dx \]
\[ \int \sin^n x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m + n} + \frac{n - 1}{m + n} \int \sin^m x \cos^{n-2} x \, dx \]
\[ \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx \]
\[ \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx \]
\[ \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx \]
\[ \int \ln x \, dx = x \ln x - x \]
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<tr>
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<tr>
<td>$y(t)$</td>
<td>$Y(p) = \int_0^\infty y(t) e^{-pt} , dt$</td>
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<tr>
<td>$1$</td>
<td>$\frac{1}{p}$</td>
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<tr>
<td>$t^r$, $r &gt; -1$</td>
<td>$\frac{\Gamma(r+1)}{p^{r+1}}$</td>
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<tr>
<td>$e^{-bt}$</td>
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<tr>
<td>$\sin nt$</td>
<td>$\frac{n}{p^2 + n^2}$</td>
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<tr>
<td>$\cos nt$</td>
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<tr>
<td>$e^{-bt} f(t)$</td>
<td>$F(p+b)$</td>
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<tr>
<td>$h(t-a)$</td>
<td>$\frac{e^{-ap}}{p}$</td>
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<td>$f(t-a) h(t-a)$</td>
<td>$e^{-ap} F(p)$</td>
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<tr>
<td>$\delta(t-a)$</td>
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<tr>
<td>$f'(t)$</td>
<td>$pF(p) - f(0)$</td>
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<tr>
<td>$f''(t)$</td>
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<tr>
<td>$\int_0^t f(t-u) , g(u) du$</td>
<td>$F(p) G(p)$</td>
</tr>
<tr>
<td>$t^n f(t)$</td>
<td>$(-1)^n \frac{d^n}{dp^n} F(p)$</td>
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