Chapter 5: Laplace Transforms

The Laplace Transform is a useful tool that is used to solve many mathematical and applied problems. In particular, the Laplace transform is a technique that can be used to solve linear constant coefficient differential equations in one or more equations. This tool is used to transform the given system of differential equations into a system of algebraic equations that are then solved simultaneously. The algebraic system obtained is usually much easier to solve than the given differential equations. Graphically,

As shown in the Figure (a) the original function \( f(t) \) (solution to the given differential equation) is transformed into a new function \( F(s) \) via a transfer function. This function \( F(s) \) is obtained from solving an algebraic equation in \( s \). Once \( F(s) \) is found we use transforms to obtain the original variable \( t \) and hence, obtain the solution to the given differential equation. This is described in Figure (b).

The Laplace transform method is used in a wide number of engineering problems, in particular, the design of control systems in electrical engineering or the vibration of mass-spring system to a unit impulse. For example, Laplace transforms can be used to determine the response of a damped mass-spring system which is initially at rest and is suddenly given a sharp hammer blow. The governing equations could be represented by

\[
y'' + 3y' + 2y = \delta(t - a)
\]

where

\[
y(0) = 0 \quad \text{and} \quad y'(0) = 0.
\]

Our usual methods cannot solve this equation due to the impulse function, \( \delta(t - a) \). However, using Laplace transforms we can solve this equation. It’s solution being

\[
y(t) = \left( e^{-(t-a)} - e^{-2(t-a)} \right) h(t - a)
\]

where \( h(t) \) is the step function.

Discussion on solving differential equations will be given in later in this chapter. However, first, we need to become familiar with the Laplace transform and its properties.
5.1 DEFINITION

Let \( f(t) \) be a function defined on the interval \([0, \infty)\). Then the Laplace Transform of a function \( f(t) \) is defined by

\[
\mathcal{L}\{f(t)\} = F(p) = \int_0^\infty e^{-pt} f(t) dt
\]

**Note:**

Here the independent variable of \( f \) is \( t \) and the transfer parameter is \( p \). This means we are transferring from the \( t \) domain to the \( p \) domain when using Laplace transforms. The parameter \( p \) can be complex. However, provided \( f(t) \) is a ‘well behaved’ function, then the Re \( \{p\} \) must be positive for the above integral to converge.

In some texts \( s \) is used as the transfer parameter instead of \( p \) although the same definition holds. That is,

\[
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt.
\]

In MATH202 both parameters may be used.

There are a variety of transform functions which can aid the mathematician or engineer to solve problems. These will not be discussed here. However, it is worth mentioning two of these transforms.

- **Fourier:**
  \[
  F\{f(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} f(t) dt
  \]

- **Two-sided Laplace:**
  \[
  \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-pt} f(t) dt
  \]

5.2 LAPLACE TRANSFORM OF FUNCTIONS

Let \( \mathcal{L}\{f(t)\} = F(p) \) and \( \mathcal{L}\{g(t)\} = G(p) \) then particular results for the Laplace transform are

\[
\mathcal{L}\{\alpha f(t)\} = \alpha \mathcal{L}\{f(t)\} \quad \text{where } \alpha \text{ a constant.}
\]

\[
\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} \quad \text{where } \alpha \text{ and } \beta \text{ are constants.}
\]

\[
\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{p}{a}\right)
\]

\[
\mathcal{L}\{e^{-at} f(t)\} = F(p + a) \quad \text{(Shift Theorem)}
\]

\[
\mathcal{L}\{f(t-a) h(t-a)\} = e^{-ap} F(p) \quad \text{(Second Shift Theorem)}
\]

\[
\mathcal{L}\{f(t) h(t-a)\} = e^{-ap} \mathcal{L}\{f(t+a)\} \quad \text{(Third Shift Theorem)}
\]

\[
\mathcal{L}\{tf(t)\} = -\frac{d}{dp} F(p).
\]

where \( a \) is a constant.
Examples

Recall that
\[ \mathcal{L}\{f(t)\} = F(p) = \int_{0}^{\infty} f(t) e^{-pt} \, dt. \]

(a) Find the Laplace transform of \( f(t) = t \).

Method

\[ \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \int_{0}^{\infty} t e^{-pt} \, dt \quad \text{replacing } f(t) \text{ by } t \]

\[ = \frac{1}{p} \int_{0}^{\infty} e^{-pt} \, dt \quad \text{using integration by parts} \]

\[ = \frac{1}{p^2}. \]

Note:
Here \( f \) is a function of the independent variable \( t \) therefore we found the Laplace transform of \( f \) with respect to the independent variable \( t \). However, not all functions will have the independent variable being \( t \). For instance, the independent variable for \( f(x) \) is \( x \). If this is the case then we simply take Laplace transforms with respect to \( x \). In all cases, the Laplace transform function \( F(p) \) will be the same.

Convention

We usually use capital letters to represent the Laplace transform of a function. For example,

\[ \mathcal{L}\{f(x)\} = F(p), \quad \mathcal{L}\{g(u)\} = G(p) \quad \text{and} \quad \mathcal{L}\{z(w)\} = Z(p). \]

For example,

If \( f(x) = x \), say, then \( \mathcal{L}\{f(x)\} = \mathcal{L}\{x\} = \frac{1}{p^2} \) or if \( g(u) = u \) then \( \mathcal{L}\{g(u)\} = \mathcal{L}\{u\} = \frac{1}{p^2} \).

A graph of the function \( f(x) = x \) and its transform (ie \( \mathcal{L}\{f(x) = x\} = F(p) = \frac{1}{p^2} \)) is shown below.
(b) Find the Laplace transform of \( f(t) = \sin t \).

**Method**

The independent variable is \( t \). Therefore, find the Laplace transform of \( f(t) \) with respect to \( t \). That is,

\[
\mathcal{L}\{\sin t\} = \int_0^\infty \sin t \, e^{-pt} \, dt
\]

replacing \( f(t) \) by \( \sin t \).

\[
= \frac{1}{s} \int_0^\infty \cos t \, e^{-pt} \, dt
\]

using integration by parts.

\[= \frac{1}{1 + \frac{p^2}{1}}.\]

*Note:* Evaluation of this integral could have been done by Integral [31].

A graph of \( f(t) = \sin t \) and its transform \( F(p) \) is shown below.

Similarly, the \( \mathcal{L}\{\cos t\} \) is

\[
\mathcal{L}\{ \cos t \} = F(p) = \int_0^\infty \cos t \, e^{-pt} \, dt
\]

replacing \( f(t) \) by \( \cos t \).

\[
= \frac{p}{p^2 + 1}
\]

using integration by parts.

A graph of \( f(t) = \cos t \) and its transform \( F(p) \) is shown below.
(c) Find the Laplace transform of \( f(t) = t + \sin t \).

**Method**

Recall that \( \mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \). Let \( f(t) = t \) and \( g(t) = \sin t \) then

\[
\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{t + \sin t\}
\]
\[
= \mathcal{L}\{t\} + \mathcal{L}\{\sin t\}
\]
\[
= \frac{1}{p^2} + \frac{1}{p^2 + 1}.
\]

(d) Find the Laplace transform of

\[
f(t) = \begin{cases} 
1, & t \geq 3 \\
0, & t < 3
\end{cases}
\]

**Method**

Recall that

\[
\mathcal{L}\{f(t)\} = F(p) = \int_0^\infty f(t) e^{-pt} \, dt.
\]

Therefore,

\[
\mathcal{L}\{f(t)\} = \int_0^3 f(t) e^{-pt} \, dt + \int_3^\infty f(t) e^{-pt} \, dt
\]
\[
= \int_3^\infty e^{-pt} \, dt \quad \text{replacing } f(t) \text{ by its definition.}
\]
\[
= \left. \frac{e^{-pt}}{-p} \right|_3^\infty = \frac{e^{-3p}}{p}.
\]

### 5.2.1 Shift Theorem

The Shift theorem is an important theorem and will be used quite frequently throughout this chapter.

Recall that

\[
\mathcal{L}\{e^{-at}f(t)\} = F(p + a)
\]

where \( a \) is an arbitrary constant.

This theorem means that if we want to find \( \mathcal{L}\{e^{-at}f(t)\} \) we simply find the Laplace transform of \( f(t) \) which is \( F(p) \) and then replace \( p \) by \( p + a \).

Graphically, this theorem is represented by \( F(p) \) shifted to the right or left depending on the sign of \( a \).
Mathematically, we can write that

\[
\mathcal{L}\{e^{-at}f(t)\} = F(p+a) = F(p)|_{p->p+a}.
\]

Also, we shall use the shift theorem in Section 5.4 to find the inverse Laplace transform of a given function of \( p \).

**Examples**

(a) Find the Laplace transform of \( f(t) = e^{-2t}\sin t \).

**Method**

\[
\mathcal{L}\{e^{-2t}\sin t\} = \int_0^\infty (e^{-2t}\sin t)e^{-pt}dt \quad \text{replacing} \ f(t) \ \text{by} \ e^{-2t}\sin t
\]

\[
= \int_0^\infty \sin t \ e^{-(2+p)t}dt \quad \text{integrating by parts}
\]

\[
= \frac{1}{(p+2)^2 + 1}.
\]

Alternatively, we can use the shift theorem. That is,

\[
\mathcal{L}\{e^{-at}f(t)\} = F(p+a) \quad \text{where} \quad F(p) = \mathcal{L}\{f(t)\}.
\]

In our example, we see that \( f(t) = \sin t \) and \( a = 2 \) then

\[
F(p) = \mathcal{L}\{f(t)\} = \frac{1}{p^2 + 1} \quad \text{from example (d)}.
\]

Therefore,

\[
F(p+a) = \frac{1}{(p+a)^2 + 1} \quad \text{where} \quad a = 2
\]

\[
= \frac{1}{(p+2)^2 + 1}.
\]

Hence,

\[
\mathcal{L}\{e^{-2t}\sin t\} = \frac{1}{(p+2)^2 + 1}.
\]

Below is a graph of the function \( f(t) = e^{-2t}\sin t \) and the transform function \( F(p+2) = \frac{1}{(p+2)^2 + 1} \).
(b) Find the Laplace transform of $e^{3t} \cos 4t$.

Method

\[
\mathcal{L} \{ e^{3t} \cos 4t \} = \int_0^\infty e^{3t} \cos 4t \, e^{-pt} \, dt
\]

replacing $f(t)$ by $e^{3t} \cos 4t$

\[= \int_0^\infty \cos(4t) \, e^{-(p-3)t} \, dt \quad \text{integrating by parts}\n\]

\[= \frac{p-3}{(p-3)^2 + 16}.
\]

Alternatively, using the shift theorem we let $f(t) = \cos 4t$ where $a = -3$ then

\[F(p) = \frac{p}{p^2 + 16} \quad \text{from Laplace tables where } n = 4.
\]

Hence,

\[F(p-3) = \frac{p-3}{(p-3)^2 + 16} \quad \text{where } a = -3.
\]

Therefore,

\[\mathcal{L} \{ e^{3t} \cos 4t \} = \frac{p-3}{(p-3)^2 + 16}.
\]

The previous two examples show alternative methods of finding the Laplace transform of functions of the form $\mathcal{L} \{ e^{-at} f(t) \}$. Each of these methods is acceptable. However, time and lengthy integration is reduced if the shift theorem and the Laplace transform tables are used.

### 5.2.2 The Convolution Theorem

The Convolution Theorem, states

\[\mathcal{L} \left\{ \int_0^t f(t-u) \, g(u) \, du \right\} = F(p) \, G(p)
\]

and is often written as

\[\mathcal{L} \{ f * g \} = F . G
\]

where $\mathcal{L} \{ f(t) \} = F(p)$ and $\mathcal{L} \{ g(t) \} = G(p)$.

Alternatively, the Convolution theorem can also be written in the form

\[\mathcal{L} \left\{ \int_0^t f(u) \, g(t-u) \, du \right\} = F(p) \, G(p)
\]

where the end result is still the same.

**Note:**

It should be noted that the given $\int_0^t f(u) \, g(t-u) \, du$ is a function of $t$ and therefore, when we take Laplace transforms we are doing it with respect to $t$.

A special case of the Convolution Theorem is the result for the Laplace transform of an integral.

\[\mathcal{L} \left\{ \int_0^t f(u) \, du \right\} = \frac{1}{p} F(p).
\]
Examples

(a) Find the Laplace transform of \( \int_0^t u \sin(t-u) \, du \).

Method

It can be easily seen that \( \int_0^t u \sin(t-u) \, du \) is a function of \( t \). Therefore, let

\[
k(t) = \int_0^t u \sin(t-u) \, du.
\]

Therefore,

\[
\mathcal{L}\{k(t)\} = \int_0^\infty k(t) e^{-pt} \, dt
\]

\[= \int_0^\infty \left( \int_0^t u \sin(t-u) \, du \right) e^{-pt} \, dt.
\]

Reversing the order of integration.

That is,

\[
\mathcal{L}\{k(t)\} = \int_0^\infty u \left( \int_u^\infty e^{-pt} \sin(t-u) \, dt \right) du
\]

as \( u \) is a constant in the inner integral

Let \( z = t - u \) then \( dz = dt \);
when \( t = u, \quad z = 0; \quad t \to \infty, \quad z \to \infty \).

\[
= \int_0^\infty u e^{-u} \left( \int_0^\infty e^{-pz} \sin(z) \, dz \right) du
\]

The inner integral is \( \mathcal{L}\{\sin z\} \). That is,

\[
\mathcal{L}\{\sin z\} = \frac{1}{p^2 + 1}.
\]

This is the \( \mathcal{L}\{u\} \) which is \( \frac{\Gamma(2)}{p^2} \).

\[
= \frac{1}{p^2 + 1} \int_0^\infty u e^{-u} \, du
\]

\[
= \frac{1}{p^2(p^2 + 1)}.
\]

Alternatively, use the Convolution theorem and let

\[
f(t-u) = \sin(t-u) \quad \text{and} \quad g(u) = u.
\]

Thus,

\[
f(t) = \sin t \quad \text{(replacing } t-u \text{ by } t \text{)} \quad \text{and} \quad g(t) = t.
\]

Taking Laplace transforms of both \( f(t) \) and \( g(t) \) with respect to \( t \) we have

\[
\mathcal{L}\{f(t)\} = F(p) = \frac{1}{p^2 + 1} \quad \text{and} \quad \mathcal{L}\{g(t)\} = G(p) = \frac{1}{p^2}.
\]

Therefore,

\[
\mathcal{L}\left\{ \int_0^t u \sin(t-u) \, du \right\} = \frac{1}{p^2 + 1} \times \frac{1}{p^2}
\]

\[
= \frac{1}{p^2(p^2 + 1)}.
\]
Note:
It can be seen in this example that using the Convolution theorem is quicker than doing the double integration. Therefore, it wise that students learn this theorem.

(b) Find \( L \left\{ \int_0^t e^{-2u} \sinh 3(t - u) \, du \right\} \).

Method

Matching the functions in the Convolution theorem we can see that

\[
 f(t - u) = \sinh 3(t - u) \quad \text{and} \quad g(u) = e^{-2u}
\]

Thus,

\[
 f(t) = \sinh 3t \quad \text{(replacing } t - u \text{ by } t \text{)} \quad \text{and} \quad g(t) = e^{-2t}.
\]

We take Laplace transforms of both \( f(t) \) and \( g(t) \) with respect to \( t \). Therefore, using the Laplace transform tables, we find that

\[
 L \{ f(t) \} = F(p) = \frac{3}{p^2 - 9} \quad \text{and} \quad L \{ g(t) \} = G(p) = \frac{1}{p + 2}.
\]

Therefore,

\[
 L \left\{ \int_0^t e^{-2u} \sinh(3(t - u)) \, du \right\} = \frac{3}{p^2 - 9} \times \frac{1}{p + 2} = \frac{3}{(p + 2)(p^2 - 9)}.
\]

(c) Find \( L \left\{ \int_0^x (x - u)^2 \sin 4u \, du \right\} \).

Here we are taking the Laplace transform with respect to \( x \).

Method

Matching the functions in the Convolution theorem, we can see that

\[
 f(x - u) = (x - u)^2 \quad \text{and} \quad g(u) = \sin 4u.
\]

Thus,

\[
 f(x) = x^2 \quad \text{(replacing } x - u \text{ by } x \text{)} \quad \text{and} \quad g(x) = \sin 4x.
\]

Taking Laplace transforms of both \( f(x) \) and \( g(x) \) with respect to \( x \) by using the Laplace transform tables, we obtain

\[
 L \{ f(x) \} = F(p) = \frac{\Gamma(3)}{p^3} \quad \text{and} \quad L \{ g(x) \} = G(p) = \frac{4}{p^2 + 16}.
\]

Therefore,

\[
 L \left\{ \int_0^x (x - u)^2 \sin u \, du \right\} = \frac{\Gamma(3)}{p^3} \times \frac{4}{p^2 + 16} = \frac{4\Gamma(3)}{p^3(p^2 + 16)} = \frac{8}{p^3(p^2 + 16)}.
\]
5.2.3 Laplace Transform of a Derivative

The Laplace transform of a derivative is:

\[
\mathcal{L}\{ f(t) \} = F(p) \\
\mathcal{L}\{ f'(t) \} = pF(p) - f(0) \\
\mathcal{L}\{ f''(t) \} = p^2 F(p) - pf(0) - f'(0) \\
\mathcal{L}\{ f'''(t) \} = p^3 F(p) - p^2 f(0) - pf'(0) - f''(0)
\]

These results can be used to solve linear differential equations, and systems of linear differential equations.

Examples

(a) Find the Laplace transform of \( \frac{dy}{dx} - 3y \) where \( y(0) = 1 \).

Method

Using the above information, we let \( \mathcal{L}\{ y(x) \} = Y(p) \) then

\[
\mathcal{L}\left\{ \frac{dy}{dx} \right\} = pY(p) - y(0) = pY(p) - 1, \quad \text{where } y(0) = 1.
\]

Therefore,

\[
\mathcal{L}\left\{ \frac{dy}{dx} - 3y \right\} = pY(p) - 1 - 3Y(p) = (p - 3)Y(p) - 1
\]

(b) Find \( \mathcal{L}\left\{ \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y \right\} \) where \( y(0) = 2 \) and \( y'(0) = -1 \).

Method

Let \( \mathcal{L}\{ y(x) \} = Y(p) \) then

\[
\mathcal{L}\left\{ \frac{dy}{dx} \right\} = pY(p) - y(0) \quad \text{and} \quad \mathcal{L}\left\{ \frac{d^2y}{dx^2} \right\} = p^2Y(p) - py(0) - y'(0).
\]

Using the given conditions we find that

\[
\mathcal{L}\left\{ \frac{dy}{dx} \right\} = pY(p) - 2 \quad \text{and} \quad \mathcal{L}\left\{ \frac{d^2y}{dx^2} \right\} = p^2Y(p) - 2p + 1.
\]

Therefore,

\[
\mathcal{L}\left\{ \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y \right\} = \mathcal{L}\left\{ \frac{d^2y}{dx^2} \right\} - \mathcal{L}\left\{ \frac{dy}{dx} \right\} + \mathcal{L}\left\{ 4y \right\} \\
= p^2Y(p) - 2p + 1 - (pY(p) - 2) + 4Y(p) \\
= (p^2 - p + 4)Y(p) - 2p + 3.
\]
5.2.4 Laplace Transform of Other Types

Recall that
\[ \mathcal{L}\{tf(t)\} = -\frac{d}{dp} F(p) \quad \text{where } \mathcal{L}\{f(t)\} = F(p). \]

**Examples**

(a) Find the Laplace transform of \( t \sinh 2t \).

**Method**

Let \( f(t) = \sinh 2t \) then
\[ F(p) = \frac{2}{p^2 - 4}. \]

From Laplace tables where \( n = 2 \).

Therefore,
\[
\mathcal{L}\{ t \sinh 2t \} = -\frac{d}{dp} \left( \frac{2}{p^2 - 4} \right) = \frac{4p}{(p^2 - 4)^2}.
\]

(b) Let \( y = y(t) \). Find the Laplace transform of \( ty'' + y' - ty \) where \( y(0) = 2 \) and \( y'(0) = 0 \).

**Method**

Let \( \mathcal{L}\{y(t)\} = Y(p) \) then
\[ \mathcal{L}\{ty\} = -\frac{d}{dp} Y(p) \quad \text{that is, } \quad \mathcal{L}\{ty\} = -Y'(p). \]

Also,
\[ \mathcal{L}\{y'\} = pY(p) - y(0) \quad \text{that is, } \quad \mathcal{L}\{y'\} = pY(p) - 2. \]

and
\[ \mathcal{L}\{ty''\} = -\frac{d}{dp} [p^2Y(p) - py(0) - y'(0)] \quad \text{that is, } \quad \mathcal{L}\{ty''\} = -p^2Y'(p) - 2pY(p) + 2. \]

Therefore,
\[
\mathcal{L}\{ty'' + y' - ty\} = \mathcal{L}\{ty''\} + \mathcal{L}\{y'\} - \mathcal{L}\{ty\}
= -p^2Y'(p) - 2pY(p) + 2 + (pY(p) - 2) - (-Y'(p))
= (1 - p^2)Y'(p) - pY(p).
\]

**Note:**

Students should become familiar with Laplace transforms, its properties and the Table of Laplace transforms which are at the end of this chapter, as well as, the inside of the back cover.
Exercise 5A

1 This question refers to the table of Basic Laplace Transforms in section 5.6.

(a) Derive each of the results on the first 8 lines of the table by using the definition of a Laplace transform.

(b) From the definition, evaluate the following Laplace transforms.

(i) \( \mathcal{L}\{t \sin nt\} \)

(ii) \( \mathcal{L}\{t \cos nt\} \)

(iii) \( \mathcal{L}\{e^t \sin nt\} \)

(iv) \( \mathcal{L}\{e^t \cos nt\} \)

(v) \( \mathcal{L}\{\sin nt - nt \cos nt\} \)

(vi) \( \mathcal{L}\{e^{3t}(\sin nt + \cos nt)\} \)

(vii) \( \mathcal{L}\{t(\sinh 3t + \cos 2t)\} \)

2 Find the Laplace transform of the following functions.

(a) \( f(t) = t^2 e^{3t} + 1 \)  
(b) \( g(t) = t^2 \sinh 3t \)

(c) \( f(x) = \cos x \sin x \)  
(d) \( g(x) = \cos^2 x \)

(e) \( g(t) = \begin{cases} t & t > 1 \\ 1 & 0 < t < 1 \end{cases} \)

(f) \( f(x) = \begin{cases} \cos x & x > \frac{\pi}{2} \\ \sin x & 0 < x < \frac{\pi}{2} \end{cases} \)

(g) \( k(t) = \int_0^t e^{-t} \sin u \, du \)

(h) \( w(t) = \int_0^t \sin 2(t - u) \sin u \, du \)

(i) \( k(t) = \int_0^t \cos(t - u) \sin 3u \, du \)

3 Verify the following results.

(i) \( \int_0^t f(t - u) \, du = \int_0^t f(u) \, du \)

(ii) \( \int_0^t f(t - u) g(u) \, du = \int_0^t f(u) g(t - u) \, du \)

4 Use the table of Laplace transforms to write down the values of the following integrals.

(a) \( \int_0^\infty e^{-5t} \, dt \)

(b) \( \int_0^\infty e^{-2t} t^7 \, dt \)

(c) \( \int_0^\infty e^{-2t} \cos 5t \, dt \)

(d) \( \int_0^\infty e^{-4t} \cosh 3t \, dt \)

(e) \( \int_0^\infty e^{-3t} \int_0^t e^{-u} \sin u \, du \, dt \)

(f) \( \int_0^\infty e^{-x} \int_0^x \sin 2(x - u) \cos u \, du \, dx \)

5 (a) Show that \( \mathcal{L}\{1 \cdot f(x)\} = \int_p^\infty F(p) \, dp. \)

(b) Hence determine the Laplace transforms of the following functions:

(i) \( \frac{\sin ax}{x} \)

(ii) \( \frac{1 - \cos ax}{x} \)

(iii) \( \frac{e^{ax} - e^{bx}}{x} \)

6 (a) Use the definition of the Laplace Transform, and integration by parts, to find the Laplace transforms of \( y'(t), \ y''(t) \).

(b) Let \( \mathcal{L}\{y(t)\} = Y(p) \). By taking Laplace transforms, find \( Y(p) \) for each of the following equations subject to the given initial conditions.

(i) \( \frac{dy}{dt} - y = e^{-t} \) where \( y(0) = 1 \).

(ii) \( 2 \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 5y = \sin 2t \)

\[ \text{where } y(0) = -1 \text{ and } y'(0) = 1. \]
5.3 LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS

In this section we will look at the Laplace transform of some important special functions.

5.3.1 Step (Heaviside) Function

Recall the step function definition:

\[
h(t - a) = \begin{cases} 
1 & t \geq a \\
0 & t < a.
\end{cases}
\]

Then the Laplace transform of \( h(t - a) \) is

\[
\mathcal{L}\{h(t-a)\} = \int_0^\infty h(t-a)e^{-px} \, dx = \int_a^\infty e^{-px} \, dx = \frac{e^{-ap}}{p}.
\]

Therefore, the inverse Laplace of \( \frac{e^{-ap}}{p} \) is \( h(t - a) \). That is,

\[
\mathcal{L}^{-1}\left\{\frac{e^{-ap}}{p}\right\} = h(t - a).
\]

Property

\[
\mathcal{L}\{f(t-a)h(t-a)\} = e^{-ap}F(p)
\]

where \( \mathcal{L}\{f(t)\} = F(p) \).

Examples

(a) Find \( \mathcal{L}\{h(t-3)\} \).

Method

Recall that \( \mathcal{L}\{h(t-a)\} = \frac{e^{-ap}}{p} \) then

\[
\mathcal{L}\{h(t-3)\} = \frac{e^{-3p}}{p}, \quad \text{where } a = 3.
\]

(b) Find \( \mathcal{L}\{th(t-2)\} \).

Method

Reshaping the given function we have

\[
th(t-2) = (t-2)h(t-2) + 2h(t-2).
\]

Therefore,

\[
\mathcal{L}\{th(t-2)\} = \mathcal{L}\{(t-2)h(t-2)\} + 2\mathcal{L}\{h(t-2)\}.
\]
Consider the first term on the right hand side of the equation. Noting the property that

\[ \mathcal{L}\{ f(t-a)h(t-a) \} = e^{-ap} F(p) \quad \text{where} \quad F(p) = \mathcal{L}\{ f(t) \} \]

then

\[ f(t-2) = t-2 \quad \text{where} \quad a = 2. \]

Hence,

\[ f(t) = t \quad \text{where} \quad t-2 \text{ is replaced by } t. \]

Therefore,

\[ \mathcal{L}\{ f(t) \} = F(p) = \mathcal{L}\{ t \} \]

\[ = \frac{\Gamma(2)}{p^2} = \frac{1}{p^2}. \]

\[ \Rightarrow \mathcal{L}\{ (t-2)h(t-2) \} = \frac{e^{-2p}}{p^2}, \quad \text{where} \quad a = 2. \]

Also,

\[ \mathcal{L}\{ 2h(t-a) \} = 2 \frac{e^{-2p}}{p}, \quad \text{where} \quad a = 2. \]

As a result we find that

\[ \mathcal{L}\{ t h(t-2) \} = \frac{e^{-2p}}{p^2} + 2 \frac{e^{-2p}}{p} = \frac{e^{-2p}}{p^2}(1 + 2p). \]

### 5.3.2 Dirac Delta (Impulse) Function

Recall that

\[ f_k(t-a) = \begin{cases} \frac{1}{k} & a < t < a+k \\ 0 & \text{otherwise.} \end{cases} \]

The limit of \( f_k(t) \) as \( k \to 0 \) is denoted by \( \delta(t-a) \). Now

\[ \mathcal{L}\{ f_k(t-a) \} = \int_0^\infty e^{-pt} f_k(t-a) \, dt \]

\[ = \int_a^{a+k} e^{-pt} \frac{1}{k} \, dt \]

\[ = \frac{1}{k} \left[ e^{-pt} \right]_a^{a+k} \]

\[ = \frac{e^{-pa} (1 - e^{-pk})}{kp}. \]

Now

\[ \lim_{k \to 0} \mathcal{L}\{ f_k(t-a) \} = \lim_{k \to 0} e^{-pa} \frac{1 - e^{-pk}}{kp} \quad \left( = \frac{0}{0} \right) \]

\[ = e^{-pa} \cdot \frac{vH}{k} \frac{e^{-pa} (pe^{-pk})}{p} \]

\[ = e^{-pa}. \]
Hence,
\[ \mathcal{L}\{\delta(t - a)\} = e^{-pa}. \]

**Example**

Find \( \mathcal{L}\{\delta(x - 2)\} \).

**Method**

\[ \mathcal{L}\{\delta(x - 2)\} = e^{-2p} \quad \text{From tables, where } a=2. \]

**Note:**

The Dirac Delta function is sometimes called the *unit impulse function*. It is not a function in the ordinary sense. The definition of the Dirac Delta function can vary depending on the text.

### 5.3.3 Error Function

Recall the definition of the error function. That is,
\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} \, du \]

then \( \mathcal{L}\{\text{erf}(t)\} \) is obtained by taking the Laplace transform with respect to \( t \). That is,

\[
\mathcal{L}\{\text{erf}(t)\} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-pt} \text{erf}(t) \, dt
\]

Reverse the order of integration.

\[
= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left( \int_0^\infty e^{-pt} \, dt \right) \, du
\]

Let \( z = u + \frac{p}{2} \) then \( dz = du \)

when \( u = 0, z = \frac{p}{2} \); when \( u = \infty, z = \infty \)

\[
= \frac{2}{\sqrt{\pi}} e^{\frac{p^2}{4}} \int_{\frac{p}{2}}^\infty \frac{e^{-z^2}}{p} \, dz
\]

That is,

\[
\mathcal{L}\{\text{erf}(t)\} = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(u^2+pu)}}{p} \, du
\]

Complete the square.

\[
= \frac{2}{\sqrt{\pi}} e^{\frac{p^2}{4}} \int_{\frac{p}{2}}^\infty \frac{e^{-z^2}}{p} \, dz
\]

Let \( z = u + \frac{p}{2} \) then \( dz = du \)

when \( u = 0, z = \frac{p}{2} \); when \( u = \infty, z = \infty \)

\[
= \frac{2}{\sqrt{\pi}} e^{\frac{p^2}{4}} \int_{\frac{p}{2}}^\infty \frac{e^{-z^2}}{p} \, dz
\]

\[
= \frac{e^{\frac{p^2}{4}} \text{erfc}(\frac{p}{2})}{p}.
\]
**Exercise 5B**

1 Define the functions \( h(t) \), \( h(t-a) \), \( \delta(t-a) \) and find their Laplace transforms.

Use the Convolution theorem to show that

\[
\mathcal{L} \left\{ \int_0^t h(t-u)x(u) \, du \right\} = \frac{X(p)}{p}.
\]

2 Find the Laplace transform of the following functions.
   (a) \( h(t-2) \)
   (b) \( h(t-2)e^{(t-2)} \)
   (c) \( h(t-2)e^t \)
   (d) \( \sin(3t) \left( h(t - \frac{\pi}{2}) - h(t) \right) \)
   (e) \( \delta(t-5) \).
   (f) \( \text{erfc}(\sqrt{t}) \)
   (g) \( \int_0^t h(t-u) \sin 3u \, du \)
   (h) \( \int_0^t h(t-u) \cos 2u \, du \)
   (i) \( e^{-t^2} \).
   (j) \( \int_0^\infty \delta(u-t) \sin 3u \, du \) for \( t > 0 \)
   (k) \( \int_0^\infty \delta(u-t) \cos 2u \, du \) for \( t > 0 \)
   (l) \( \int_0^t h(t-u-2) \sin 3u \, du \)
   (m) \( \int_0^t h(t-u-3) \cos 2u \, du \)

3 Let \( \mathcal{L}\{y(t)\} = Y(p) \). Find \( Y(p) \) if
   (a) \( \frac{dy}{dx} - y = \delta(x-1) \)
   (b) \( \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = \delta(t-3) \)

4 Use transform definitions, and the evaluation of a suitable double integral, to calculate the Laplace transform of the following integrals.
   (i) \( \mathcal{L} \left\{ \int_0^t x(u) \, du \right\} \)
   (ii) \( \mathcal{L} \left\{ \int_0^t f(t-u) g(u) \, du \right\} \)
      (Convolution integral).

5 Evaluate \( \int_0^x \text{erf}(\sqrt{x-t}) \text{erf}(\sqrt{t}) \, dt \).

6 Prove the following results.
   (a) \( \mathcal{L}\{\delta(x)\} = 1 \)
   (b) \( \mathcal{L}\{\delta(x-a)\} = e^{-ap} , \quad a \geq 0 \)

7 (a) Show that the Laplace transform of

\[
f(x-a)h(x-a) \quad \text{is} \quad e^{-ap}F(p).\]

(b) Hence find the functions whose transforms are given below.
   (i) \( \frac{e^{-ap}}{p^2} , \quad a > 0 \)
   (ii) \( \frac{e^{-ap}}{p^2 + 1} \).

5.4 INVERSE LAPLACE TRANSFORM OF A FUNCTION

The inversion formula for the Laplace transform is

\[
f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} F(p) \, dp
\]

where \( \gamma \) is called the Bromwich Line.
We usually denote the inverse Laplace transform by \( \mathcal{L}^{-1} \). That is,

\[
f(t) = \mathcal{L}^{-1}\{F(p)\}
\]

where \( f(t) \) is the original function.

**Note:**
The assumption here is that the independent variable of the original function \( f \) is \( t \) but this not necessary the case.

In this work, we will rely on using the table in section 5.6 to invert many transforms as the inversion formula requires complex analysis methods. However, we will use the Laplace transform properties, the table of Laplace transforms and the Convolution theorem to aid in determining the inverse Laplace transform of a function.

### 5.4.1 Properties of the Inverse Laplace Transform

Let \( f(t) = \mathcal{L}^{-1}\{F(p)\} \) and \( g(t) = \mathcal{L}^{-1}\{G(p)\} \) and let \( \alpha \) and \( \beta \) be two arbitrary constants then

\[
\mathcal{L}\mathcal{L}^{-1} = \mathcal{L}^{-1}\mathcal{L} = I \quad \text{where} \quad I \text{ is the identity operator.}
\]

\[
\mathcal{L}^{-1}\{\alpha F(p)\} = \alpha \mathcal{L}^{-1}\{F(p)\}.
\]

\[
\mathcal{L}^{-1}\{\alpha F(p) + \beta G(p)\} = \alpha \mathcal{L}^{-1}\{F(p)\} + \beta \mathcal{L}^{-1}\{G(p)\}
\]

\[
\mathcal{L}^{-1}\{F(p + a)\} = e^{-at}f(t) \quad \text{(Shift Theorem)}
\]

### 5.4.2 Techniques for Finding the Inverse Laplace Transform

There are various techniques to finding the inverse Laplace transform of a function. We shall specifically look at four techniques.

**Technique 1: Standard Tables**

**Examples**

(a) Find the inverse Laplace transform of \( F(p) = \frac{1}{p^2} \).

**Method**

From the tables it can be seen that

\[
\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{p^{r+1}} \quad \text{for} \quad r > -1.
\]

Let \( \mathcal{L}^{-1}\{F(p)\} = f(t) \) then

\[
f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{p^{r+1}} \right\} = \frac{t^r}{\Gamma(r+1)}.
\]
In comparison to our example, it can be seen that \( r + 1 = 2 \). Therefore, \( r = 1 \) and

\[
f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{p^2} \right\} = \frac{t}{\Gamma(2)} = t
\]

(b) Find the inverse Laplace transform of \( \frac{1}{p^2 + 4} \).

**Method**

From the tables it can be seen that

\[
\mathcal{L}\{\sin 2t\} = \frac{2}{p^2 + 4}, \quad \text{where} \quad n = 2.
\]

Therefore,

\[
\mathcal{L}^{-1}\left\{ \frac{1}{p^2 + 4} \right\} = \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{2}{p^2 + 4} \right\} = \frac{1}{2} \sin 2t.
\]

(c) Find the inverse Laplace transform of \( \frac{1}{p - 3} \).

**Method**

From the tables it can be seen that

\[
\mathcal{L}\{e^{bt}\} = \frac{1}{p - b}, \quad \text{where} \quad b = -3.
\]

then

\[
\mathcal{L}^{-1}\left\{ \frac{1}{p - 3} \right\} = e^{3t}.
\]

**Note:**

The assumption in these examples is that \( \mathcal{L}^{-1}\{F(p)\} = f(t) \). That is, the original function \( f \) is a function of the independent variable \( t \) but this is not necessarily the case. We can assume any independent variable if our problem does not specify.

**Technique 2: Rational Functions**

If \( F(p) \) is a rational function of \( p \) then we can use partial fractions to simplify the rational function and tables to determine \( \mathcal{L}^{-1}\{F(p)\} \).

**Examples**

(a) Find the inverse Laplace transform of \( \frac{1}{p(p - 1)} \).

**Method**

We can see that

\[
\frac{1}{p(p - 1)} = \frac{A}{p} + \frac{B}{p - 1}
\]

where \( A \) and \( B \) need to be determined by the usual partial fraction technique. It is easily shown that \( A = -1 \) and \( B = 1 \). Therefore,
\[ \mathcal{L}^{-1}\left\{ \frac{1}{p(p-1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{-1}{p} + \frac{1}{p-1} \right\} \]
\[ = \mathcal{L}^{-1}\left\{ \frac{-1}{p} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{p-1} \right\} \]
\[ = -1 + e^t, \quad \text{using tables.} \]

(b) Find \( \mathcal{L}^{-1}\left\{ \frac{6}{p^2(p^2 + 9)} \right\} \).

**Method**

Using partial fractions we have \( \frac{6}{p^2(p^2 + 9)} = \frac{A}{p} + \frac{B}{p^2} + \frac{Cp + D}{p^2 + 9} \) where \( A, B, C \) and \( D \) need to be determined by the usual partial fraction method. It can be shown that \( A = 0, \ B = \frac{2}{3}, \ C = 0 \) and \( D = -\frac{2}{3} \). Therefore,

\[ \mathcal{L}^{-1}\left\{ \frac{6}{p^2(p^2 + 9)} \right\} = \mathcal{L}^{-1}\left\{ \frac{2}{p^2} - \frac{2}{p^2 + 9} \right\} \]
\[ = \frac{2}{3} \mathcal{L}^{-1}\left\{ \frac{1}{p^2} \right\} - \frac{2}{3} \mathcal{L}^{-1}\left\{ \frac{1}{p^2 + 9} \right\} \]
\[ = \frac{2}{3} \frac{t}{\Gamma(2)} - \frac{2}{9} \mathcal{L}^{-1}\left\{ \frac{3}{p^2 + 9} \right\} \quad \text{using tables} \]
\[ = \frac{2}{3} t - \frac{2}{9} \sin 3t, \quad \text{using tables.} \]

**Technique 3: Shift Theorem**

(a) Find the inverse Laplace transform of \( \frac{1}{(p + 1)^2} \).

**Method**

Recall the Shift theorem. That is,

\[ \mathcal{L}\left\{ e^{-at} f(t) \right\} = F(p + a) \quad \text{or} \quad e^{-at} f(t) = \mathcal{L}^{-1}\{F(p + a)\}. \]

That is,

\[ \mathcal{L}^{-1}\{F(p + a)\} = e^{-at} \mathcal{L}^{-1}\{F(p)\}. \]

In our example we let

\[ F(p + 1) = \frac{1}{(p + 1)^2}, \quad \text{where} \quad a = 1. \]

Therefore,

\[ F(p) = \frac{1}{p^2}. \]

Now

\[ \mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{ \frac{1}{p^2} \right\} \]
\[ = \frac{t}{\Gamma(2)}. \quad \text{From tables where } r + 1 = 2. \]
Hence,
\[ \mathcal{L}^{-1} \left\{ \frac{1}{(p+1)^2} \right\} = e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{p^2} \right\} = e^{-t} \frac{t}{\Gamma(2)} = te^{-t}. \]

(b) Find the inverse Laplace transform of \( \frac{1}{(p-3)^2+4} \).

**Method**

Here we let
\[ F(p - 3) = \frac{1}{(p-3)^2+4}, \quad \text{where } a = -3. \]
then
\[ F(p) = \frac{1}{p^2 + 4}. \]

Recall that
\[ \mathcal{L} \{ \sin 2t \} = \frac{2}{p^2 + 4}, \quad \text{where } n = 2. \]
then
\[ \mathcal{L}^{-1} \{ F(p) \} = \mathcal{L}^{-1} \left\{ \frac{1}{p^2 + 4} \right\} = \frac{1}{2} \sin 2t. \]

Therefore,
\[ \mathcal{L}^{-1} \left\{ \frac{1}{(p-3)^2 + 4} \right\} = e^{3t} \mathcal{L}^{-1} \left\{ \frac{1}{p^2 + 4} \right\} = e^{3t} \times \frac{1}{2} \sin 2t = \frac{e^{3t}}{2} \sin 2t. \]

(c) Find the inverse Laplace transform of \( \frac{1}{p^2 + 4p + 1} \).

**Method**

In this example we will have to complete the square before being able to use the shift theorem. That is,
\[ \frac{1}{p^2 + 4p + 1} = \frac{1}{(p + 2)^2 - 3}. \]

Therefore, let
\[ F(p + 2) = \frac{1}{(p + 2)^2 - 3}, \quad \text{then } F(p) = \frac{1}{p^2 - 3}. \]

Form the Laplace transform tables we see that
\[ \mathcal{L}^{-1} \left\{ \frac{1}{p^2 - 3} \right\} = \frac{1}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{1}{\sqrt{3}} \sinh(\sqrt{3}t). \]
Therefore,

\[ \mathcal{L}^{-1} \left\{ \frac{1}{(p+2)^2 - 3} \right\} = \frac{e^{-2t}}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{e^{-2t}}{\sqrt{3}} \sinh(\sqrt{3} t). \]

(d) Find the inverse Laplace transform of \( \frac{p}{p^2 + 4p + 1} \).

**Method**

Once again we complete the square in the denominator before using the shift theorem. That is,

\[ \frac{p}{p^2 + 4p + 1} = \frac{p}{(p+2)^2 - 3}. \]

The right hand side is not in the form of \( \mathcal{F}(p+a) \) as yet. This is due to the numerator being only a function of \( p \). Therefore, we manipulate the rational function so that we do obtain the this form. That is,

\[
\frac{p}{p^2 + 4p + 1} = \frac{p}{(p+2)^2 - 3} = \frac{p+2}{(p+2)^2 - 3} - \frac{2}{(p+2)^2 - 3} = F_1(p+2) - F_2(p+2).
\]

Now

\[ F_1(p+2) = \frac{p+2}{(p+2)^2 - 3} \quad \text{then} \quad F_1(p) = \frac{p}{p^2 - 3}. \]

From the Laplace transform tables we see that

\[ \mathcal{L}^{-1} \{ F_1(p) \} = \mathcal{L}^{-1} \left\{ \frac{p}{p^2 - 3} \right\} = \cosh \sqrt{3} t. \]

Therefore,

\[ \mathcal{L}^{-1} \left\{ \frac{p + 2}{(p + 2)^2 - 3} \right\} = e^{-2t} \cosh \sqrt{3} t. \]

Consider \( F_2(p+2) \). That is,

\[ F_2(p+2) = \frac{2}{(p+2)^2 - 3} \quad \text{then} \quad F_2(p) = \frac{2}{p^2 - 3}. \]

From the Laplace transform tables we see that

\[ \mathcal{L}^{-1} \{ F_2(p) \} = \mathcal{L}^{-1} \left\{ \frac{2}{p^2 - 3} \right\} = \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{2}{\sqrt{3}} \sinh \sqrt{3} t. \]
Therefore,

\[ \mathcal{L}^{-1} \left\{ \frac{1}{(p + 2)^2 - 3} \right\} = e^{-2t} \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} = \frac{2e^{-2t}}{\sqrt{3}} \sinh \sqrt{3} t. \]

Hence,

\[ \mathcal{L}^{-1} \left\{ \frac{p}{p^2 + 4p + 1} \right\} = e^{-2t} \cosh \sqrt{3} t - \frac{2e^{-2t}}{\sqrt{3}} \sinh \sqrt{3} t. \]

(c) Find \( \mathcal{L}^{-1} \left\{ \frac{1}{p(p^2 + 4p + 1)} \right\} \).

**Method**

Using partial fractions it is found that

\[ \frac{1}{p(p^2 + 4p + 1)} = \frac{1}{p} - \frac{p + 4}{p^2 + 4p + 1}. \]

Therefore,

\[ \mathcal{L}^{-1} \left\{ \frac{1}{p(p^2 + 4p + 1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{p} - \frac{p + 4}{p^2 + 4p + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{p} \right\} - \mathcal{L}^{-1} \left\{ \frac{p + 4}{p^2 + 4p + 1} \right\}. \]

Now

\[ \mathcal{L}^{-1} \left\{ \frac{1}{p} \right\} = 1. \]

The second term requires to be put into a form so that we can use one of our inverse Laplace transform techniques. This can be done by completing the square and adding and subtracting constants. That is,

\[ \frac{p + 4}{p^2 + 4p + 1} = \frac{p + 4}{(p + 2)^2 - 3} \]

By completing the square,

\[ = \frac{p + 2 + 2}{(p + 2)^2 - 3}, \]

replacing \( p + 4 \) by \((p + 2) + 2\).

\[ = \frac{p + 2}{(p + 2)^2 - 3} + \frac{2}{(p + 2)^2 - 3}. \]

Consider the first term on the right hand side and let \( F(p + 2) = \frac{p + 2}{(p + 2)^2 - 3} \) then \( F(p) = \frac{p}{p^2 - 3} \).

Therefore,

\[ \mathcal{L}^{-1} \left\{ F(p) \right\} = \mathcal{L}^{-1} \left\{ \frac{p}{p^2 - 3} \right\} \quad \text{Using tables where } n = \sqrt{3}. \]

\[ = \cosh(\sqrt{3} t). \]

Consider the second term on the right hand side and let \( F(p + 2) = \frac{2}{(p + 2)^2 - 3} \) then \( F(p) = \frac{2}{p^2 - 3} \).

Therefore,

\[ \mathcal{L}^{-1} \left\{ F(p) \right\} = \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{p^2 - 3} \right\} \quad \text{Using tables where } n = \sqrt{3}. \]

\[ = \frac{2}{\sqrt{3}} \sinh(\sqrt{3} t). \]

Therefore,

\[ \mathcal{L}^{-1} \left\{ \frac{1}{p(p^2 + 4p + 1)} \right\} = 1 + \left( \cosh(\sqrt{3} t) + \frac{2}{\sqrt{3}} \sinh(\sqrt{3} t) \right) e^{-2t}. \]
Technique 4: Convolution Theorem

Taking the Laplace transform of a product of two functions may be easily obtained by using the convolution theorem rather than the previous techniques. This will depend on the given function. We shall solve some of the previous examples using this technique.

Recall that
\[
\mathcal{L}\left\{ \int_0^t f(t-u) g(u) \, du \right\} = F(p) G(p)
\]
or that
\[
\mathcal{L}\left\{ \int_0^t f(u) g(t-u) \, du \right\} = F(p) G(p)
\]
then
\[
\mathcal{L}^{-1}\{F(p) G(p)\} = \int_0^t f(t-u) g(u) \, du.
\]

Examples

(a) Find \( \mathcal{L}^{-1}\left\{ \frac{1}{p(p-1)} \right\} \).

Method

We can see that
\[
\frac{1}{p(p-1)} = \frac{1}{p} \times \frac{1}{p-1}.
\]
Therefore, let
\[
F(p) = \frac{1}{p} \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{ \frac{1}{p} \right\} = 1
\]
and
\[
G(p) = \frac{1}{p-1} \quad \Rightarrow \quad g(t) = \mathcal{L}^{-1}\{G(p)\} = \mathcal{L}^{-1}\left\{ \frac{1}{p-1} \right\} = e^t.
\]
Now
\[
f(t-u) = 1, \quad \text{Replacing } t \text{ bt } t-u.
\]
and
\[
g(u) = e^u, \quad \text{Replacing } t \text{ bt } u.
\]
Using the Convolution theorem we have
\[
\mathcal{L}^{-1}\left\{ \frac{1}{p(p-1)} \right\} = \int_0^t 1 \cdot e^u \, du \quad \text{substituting for } f(t-u) \text{ and } g(u)
\]
into the Convolution theorem.
\[
= e^u\bigg|_0^t
\]
\[
=e^t - 1.
\]
This is the same answer as that obtained in Example (a) using Technique two.

(b) Find \( \mathcal{L}^{-1}\left\{ \frac{6}{p^2(p^2 + 9)} \right\} \).

Method

We can see that
\[
\frac{6}{p^2(p^2 + 9)} = \frac{2}{p^2} \times \frac{3}{p^2 + 9}.
\]
Therefore, let

\[ F(p) = \frac{2}{p^2} \implies f(t) = \mathcal{L}^{-1}\{F(p)\} = \mathcal{L}^{-1}\left\{\frac{2}{p^2}\right\} = 2t \]

and

\[ G(p) = \frac{3}{p^2 + 9} \implies g(t) = \mathcal{L}^{-1}\{G(p)\} = \mathcal{L}^{-1}\left\{\frac{3}{p^2 + 9}\right\} = \sin 3t. \]

Now

\[ f(t - u) = 2(t - u), \quad \text{Replacing } t \text{ by } t - u. \]

and

\[ g(u) = \sin 3u, \quad \text{Replacing } t \text{ by } u. \]

Using the Convolution theorem we have that

\[ \mathcal{L}^{-1}\left\{\frac{6}{p^2(p^2 + 9)}\right\} = \int_0^t 2(t - u) \sin 3u \, du \]

substituting for \( f(t - u) \) and \( g(u) \) into the Convolution theorem.

\[ = 2 \int_0^t (t - u) \sin 3u \, du, \quad \text{Integrate by parts.} \]

\[ = 2 \left\{ \frac{(t - u) \cos 3u}{-3} \bigg|_0^t - \int_0^t (-1) \frac{\cos 3u}{-3} \, du \right\} \]

\[ = 2 \left\{ \frac{t}{3} - \frac{\sin 3u}{9} \bigg|_0^t \right\} \]

\[ = \frac{2}{3} t - \frac{2 \sin 3t}{9}. \]

This is the same as that obtained in Example (b) using Technique two.

### 5.5 INVERSE LAPLACE TRANSFORM OF SPECIAL FUNCTIONS

#### Examples

(a) Find \( \mathcal{L}^{-1}\left\{\frac{e^{-3p}}{p}\right\} \).

**Method**

Using Laplace transform tables, we find that

\[ \mathcal{L}^{-1}\left\{\frac{e^{-3p}}{p}\right\} = h(t - 3), \quad \text{where } a = 3. \]

(b) Find \( \mathcal{L}^{-1}\left\{\frac{e^{-4p}}{p^2}\right\} \).

**Method**

Recall that

\[ \mathcal{L}\{f(t - a)h(t - a)\} = e^{-ap}F(p) \implies \mathcal{L}^{-1}\{e^{-ap}F(p)\} = f(t - a)h(t - a), \]

where \( \mathcal{L}\{f(t)\} = F(p) \).
Now\[\frac{e^{-4p}}{p^2} = e^{-4p} \times \frac{1}{p^2};\]Therefore, let \(F(p) = \frac{1}{p^2}\) and \(a = 4\).

Hence,\[f(t) = \mathcal{L}^{-1}\{F(p)\} = \frac{t}{1(2)} = t\quad \Rightarrow \quad f(t - 4) = t - 4.\]

So that\[\mathcal{L}^{-1}\left\{\frac{e^{-4p}}{p^2}\right\} = (t - 4)h(t - 4).\]

(c) Find \(\mathcal{L}^{-1}\left\{\frac{pe^{-p}}{p^2 + 1}\right\}\).

\textit{Method}\n
\[\frac{pe^{-p}}{p^2 + 1} = e^{-p} \times \frac{p}{p^2 + 1};\]Therefore, let \(F(p) = \frac{p}{p^2 + 1}\) and \(a = 1\).

Hence,\[f(t) = \mathcal{L}^{-1}\{F(p)\} = \cos t \quad \Rightarrow \quad f(t - 1) = \cos(t - 1).\]

So that\[\mathcal{L}^{-1}\left\{\frac{pe^{-p}}{p^2 + 1}\right\} = \cos(t - 1) \cdot h(t - 1).\]

(d) Find \(\mathcal{L}^{-1}\{e^{-2p}\}\).

\textit{Method}\n
Refer to the Laplace transform tables
\[\mathcal{L}^{-1}\{e^{-2p}\} = \delta(t - 2) \quad \text{where} \quad a = 2.\]

(e) Find \(\mathcal{L}^{-1}\left\{\frac{e^{-2p}}{(p + 1)^2}\right\}\).

\textit{Method}\n
\[\frac{e^{-2p}}{(p + 1)^2} = e^{-2p} \times \frac{1}{(p + 1)^2};\]Therefore, let \(F(p) = \frac{1}{(p + 1)^2}\) and \(a = 2\).

Now\[\mathcal{L}^{-1}\left\{\frac{1}{(p + 1)^2}\right\} = e^{-t} \mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} \quad \text{where} \quad r + 1 = 2 \quad \text{and} \quad b = 1\]
\[= e^{-t} \frac{t}{\Gamma(2)},\]using tables.
\[= te^{-t}.\]

Therefore,\[f(t) = \mathcal{L}^{-1}\{F(p)\} = te^{-t} \quad \Rightarrow \quad f(t - 2) = (t - 2)e^{-(t-2)}.\]

So that\[\mathcal{L}^{-1}\left\{\frac{e^{-2p}}{(p + 1)^2}\right\} = (t - 2)e^{-(t-2)} \cdot h(t - 2).\]
Exercise 5C

1 Use partial fractions or the convolution theorem and the Table of Laplace Transforms, to find functions of \( t \) which have the following Laplace transforms.

\[
\begin{align*}
(a) & \quad \frac{2}{3p + 2} \\
(b) & \quad \frac{p + 2}{2p^2 + p - 1} \\
(c) & \quad \frac{1}{p(p^2 + 4)} \\
(d) & \quad \frac{p^2}{(p + 3)^3} \\
(e) & \quad \frac{1}{p^2(p^2 + 1)(p^2 + 4)} \\
(f) & \quad \frac{7p - 13}{p^2(p^2 + 6p + 13)} \\
(g) & \quad \frac{1}{p^2(p^2 - 4)} \\
(h) & \quad \frac{1}{p^2(p - 1)^2} \\
(i) & \quad \frac{2p}{p^2 + 2p + 5} \\
(j) & \quad \frac{p - 2}{p^2(p^2 + 4)}
\end{align*}
\]

2 (a) Show that the Laplace transform of \( f(x - a) \cdot h(x - a) \) is \( e^{-ap}F(p) \).

(b) Hence find the functions whose transforms are given below.

\[
\begin{align*}
(i) & \quad \frac{e^{-ap}}{p^2}, \quad a > 0 \\
(ii) & \quad \frac{e^{-\pi p}}{p^2 + 1} \\
(iii) & \quad \frac{e^{-4p}}{p^2 + 2p + 5} \\
(iv) & \quad \frac{e^{-3p}}{p(p + 2)}
\end{align*}
\]

5.6 INTEGRAL AND DIFFERENTIAL EQUATIONS

Laplace transforms can be used to solve a both integral and linear differential equations. Consider the following initial value problem (IVP)

\[
ay'' + by'' + cy = g(t)
\]

subject to \( y(0) = \alpha \) and \( y'(0) = \beta \).

Here the given differential equation is a constant co-efficient non-homogeneous differential equation with given initial conditions. That is, \( a, b \) and \( c \) are constants.

The procedure is to transform the given differential equation to an algebraic equation which in turn may be easily solved. This procedure is shown diagramatically below.

Taking transforms of both sides of the given differential equation we have

\[
a(p^2Y(p) - p y(0) - y'(0)) + b(pY(p) - y(0)) + cY(p) = G(p).
\]
Applying the given initial conditions and solving for $Y(p)$, it is found that

$$Y(p) = \frac{1}{a^2 p^2 + bp + c} (\alpha + \beta + G(p)).$$

This is can be represented in the form

$$Y(p) = \frac{1}{P(p)} (Q(p) + G(p))$$

where

$$P(p) = a^2 p^2 + bp + c, \quad Q(p) = \alpha + \beta \quad \text{and} \quad G(p) = \mathcal{L}\{g(t)\}.$$  

In this manner we have separated the effects of the initial conditions and those due to the input function $g(t)$.

The reciprocal of $P(p)$, that is, $\frac{1}{P(p)}$ is called the transfer function of the given equation (system).

$P(p) = 0$ is the auxiliary/characteristic equation of the differential equation using normal analytic methods.

Upon taking inverse Laplace transforms we have that

$$g(t) = \mathcal{L}^{-1}\left\{\frac{Q(p)}{P(p)}\right\} + \mathcal{L}^{-1}\left\{\frac{G(p)}{P(p)}\right\}.$$  

If $g(t) = 0$ then the solution $\mathcal{L}^{-1}\left\{\frac{Q(p)}{P(p)}\right\}$ of the problem is called the zero input response and when the initial conditions are all zero, the solution $\mathcal{L}^{-1}\left\{\frac{G(p)}{P(p)}\right\}$ is called the zero state response.

**Examples**

(a) Solve the following differential equation

$$y'' - y' - 2y = e^t$$

subject to $y(0) = 0$ and $y'(0) = 0$.

**Method**

Take Laplace transforms of the differential equation with respect to $t$. That is,

$$\mathcal{L}\{y'' - y' - 2y = e^t\}$$

gives

$$p^2 Y(p) - pg(0) - y'(0) - (pY(p) - y(0)) - 2Y(p) = \frac{1}{p - 1}.$$  

Using the given initial conditions and simplifying, we have

$$Y(p) = \frac{1}{p - 1} \frac{1}{p^2 - p - 2} = \frac{1}{(p^2 - 1)(p - 2)}.$$  

At this stage we can use partial fractions or the Convolution theorem. In this, example we shall use the Convolution theorem.
Let \( F(p) = \frac{1}{(p - 2)} \) and \( G(p) = \frac{1}{(p^2 - 1)} \).

Note: The choice of \( F(p) \) and \( G(p) \) is arbitrary.

Now
\[
\mathcal{L}^{-1} \{ F(p)G(p) \} = \mathcal{L}^{-1} \left\{ \frac{1}{(p - 2)} \right\} \times \mathcal{L}^{-1} \left\{ \frac{1}{(p^2 - 1)} \right\}
\]
then
\[
f(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(p - 2)} \right\} = e^{2t}
\]
and
\[
g(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(p^2 - 1)} \right\} = \sinh t.
\]

Therefore,
\[
f(t - u) = e^{2(t - u)} \quad \text{and} \quad g(u) = \sinh u.
\]

Using the Convolution theorem we find that the solution to the differential equation is
\[
y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(p - 1)(p^2 - p - 2)} \right\}
= \int_0^t e^{2(t - u)} \sinh u \, du
= -\frac{e^t}{2} + \frac{e^{2t}}{3} + \frac{e^{-t}}{6}.
\]

Our solution can be checked by using \( D \) operator techniques. We find that the general solution to the differential equation is
\[
y(t) = Ae^{-t} + Be^{2t} + \frac{e^t}{2}.
\]

Using the initial conditions we find that
\[
y = \frac{1}{6}e^{-t} + \frac{1}{3}e^{2t} - \frac{e^t}{2}.
\]

(b) Solve, using Laplace transforms, the following differential equation
\[
\frac{dy}{dx} + 2y = \cos 2x,
\]
subject to \( y(0) = 1 \).

Method

Take Laplace transforms of the differential equation with respect to \( x \). That is,
\[
\mathcal{L} \left\{ \frac{dy}{dx} + 2y = \cos 2x \right\}
\]
gives
\[
pY(p) - y(0) + 2Y(p) = \frac{p}{p^2 + 4}.
\]

Using the given initial conditions and simplifying, we have
\[
Y(p) = \frac{1}{p + 2} + \frac{p}{(p + 2)(p^2 + 4)}.
\]
Now
\[ \mathcal{L}^{-1}\left\{ \frac{1}{p + 2} \right\} = e^{-2x} \]
and
\[ \mathcal{L}^{-1}\left\{ \frac{p}{(p + 2)(p^2 + 4)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{p + 2} \times \frac{p}{(p^2 + 4)} \right\}. \]

Let \( F(p) = \frac{1}{p + 2} \) and \( G(p) = \frac{p}{p^2 + 4} \).

Using the Laplace transform tables we find that
\[ \mathcal{L}^{-1}\left\{ \frac{1}{p + 2} \right\} = e^{-2x} \] and \[ \mathcal{L}^{-1}\left\{ \frac{p}{(p^2 + 4)} \right\} = \cos 2x. \]

Then
\[ f(x) = e^{-2x} \quad g(x) = \cos 2x. \]

Hence,
\[ f(x - u) = e^{-2(x - u)} \quad g(u) = \cos 2u. \]

Therefore, the Convolution theorem produces
\[ \mathcal{L}^{-1}\left\{ \frac{p}{(p + 2)(p^2 + 4)} \right\} = \int_0^x e^{-2(x - u)} \cos 2u \, du \]
\[ = e^{-2x} \int_0^x e^{2u} \cos 2u \, du. \]

The integral on the right hand side of the above equation is a recurring integral after integrating by parts. The working is as follows.

Let
\[ I = e^{-2x} \int_0^x e^{2u} \cos 2u \, du \quad \text{integrate by parts} \]
\[ = e^{-2x} \left\{ \frac{e^{2u} \cos 2u}{2} \bigg|_0^x + \int_0^x e^{2u} \sin 2u \, du \right\} \quad \text{integrate by parts} \]
\[ = e^{-2x} \left\{ \frac{e^{2x} \cos 2x - 1}{2} + \frac{e^{2x} \sin 2u}{2} \bigg|_0^x \right\} - I. \]

That is,
\[ 2I = e^{-2x} \left\{ \frac{e^{2x} \cos 2x - 1}{2} + \frac{e^{2x} \sin 2x}{2} \right\}. \]

Therefore,
\[ I = e^{-2x} \left\{ \frac{e^{2x} \cos 2x - 1}{4} + \frac{e^{2x} \sin 2x}{4} \right\} \]
\[ = \frac{\sin 2x + \cos 2x - e^{-2x}}{4}. \]

and
\[ \mathcal{L}^{-1}\left\{ \frac{p}{(p + 2)(p^2 + 4)} \right\} = \frac{\sin 2x + \cos 2x - e^{-2x}}{4}. \]

Hence, the solution to the given differential equation subject to the given initial condition is
\[ y(x) = e^{-2x} + \frac{\sin 2x + \cos 2x - e^{-2x}}{4}. \]

(c) Solve the integral equation for \( y(t) \) where
\[ y(t) = 1 + \int_0^t y(u) \, du. \]
Method

Let $\mathcal{L}\{y(t)\} = Y(p)$ then the Laplace transform of the integral equation becomes

$$Y(p) = \frac{1}{p} + \mathcal{L}\left\{\int_0^t y(u) \, du\right\}.$$ 

Now

$$\mathcal{L}\left\{\int_0^t y(u) \, du\right\} = \mathcal{L}\left\{\int_0^t 1 \cdot y(u) \, du\right\} = \mathcal{L}\{1\} \times \mathcal{L}\{y(u)\} = \frac{1}{p} Y(p).$$

Therefore, we have

$$Y(p) = \frac{1}{p} + \frac{1}{p} Y(p).$$

Upon solving for $Y(p)$ we obtain that

$$Y(p) = \frac{1}{p - 1}.$$ 

Finding the inverse Laplace transform of $Y(p)$ we have that

$$y(t) = \mathcal{L}^{-1}\{Y(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{p - 1}\right\} = e^t.$$

(d) Solve the integro-differential equation for $f(t)$ where

$$f'(t) = \sinh 2t - 4 \int_0^t f(u) \cosh 2(t - u) \, du$$ 

subject to $f(0) = 1$.

Method

Let $\mathcal{L}\{f(t)\} = F(p)$ then

$$\mathcal{L}\{f'(t)\} = pF(p) - f(0) = pF(p) - 1$$

using the given initial condition and

$$\mathcal{L}\left\{\int_0^t f(u) \cosh 2(t - u) \, du\right\} = F(p) \cdot \frac{p}{p^2 - 4}$$

where

$$g(t - u) = \cosh 2(t - u) \implies g(u) = \cosh 2u \implies G(p) = \mathcal{L}\{g(u)\} = \frac{p}{p^2 - 4}.$$ 

Also, $\mathcal{L}\{\sinh 2t\} = \frac{2}{p^2 - 4}$. Therefore,

$$pF(p) - 1 = \frac{2}{p^2 - 4} - 4 \left\{ F(p) \cdot \frac{p}{p^2 - 4} \right\}.$$ 

That is,

$$\left(p + \frac{4p}{p^2 - 4}\right) F(p) = 1 - \frac{2}{p^2 - 4}.$$
Upon simplifying we have
\[
\frac{p^3}{p^2 - 4} F(p) = 1 + \frac{2}{p^2 - 4}.
\]

Upon solving for \( F(p) \) we have
\[
F(p) = \frac{1}{p} - \frac{2}{p^3}.
\]

Therefore, using the Laplace transform tables we find that
\[
f(t) = 1 - t^2.
\]

**Exercise 5D**

1. Use Laplace transforms to solve the following differential equations.
   (a) \( x'' + 7x' + 12x = 12 \)
      where \( x(0) = 0, \quad x'(0) = 5 \)
   (b) \( x'' + 3x' + 2x = 2e^{-2t} \)
      where \( x(0) = 1, \quad x'(0) = -5 \)
   (c) \( x'' + 2x' + 2x = 0 \)
      where \( x(0) = 1, \quad x'(0) = 0 \)
   (d) \( x'' + x' - 2x = \cos t + 2\sin t \)
      where \( x(0) = 1, \quad x'(0) = 1 \)

2. Use the Convolution theorem to find the inverse transforms of the following functions.
   (a) \( \frac{1}{p(p^2 + 9)} \)
   (b) \( \frac{p}{(p^2 + 1)(p^2 + 4)} \)
   (c) \( \frac{1}{p^2(p^2 - 4)} \)
   (d) \( \frac{1}{(p^2 + n^2)^2} \)

   (Hint: \( \frac{1}{(p^2 + n^2)^2} = \frac{1}{p^2 + n^2} \times \frac{1}{p^2 + n^2} \)).
   (e) \( \frac{1}{(p^2 - 4)^2} \)
   (f) \( \frac{s}{(p^2 - 4)^2} \)

3. Given that
   \( \mathcal{L} \{ e^{-bt} x(t) \} = X(p + b), \) (Shift Theorem)
   \( \mathcal{L} \{ t^n x(t) \} = (-1)^n \frac{d^n}{dp^n} X(p), \)
   when \( n \) is an integer.

   Find the following Laplace transforms:
   (a) \( t^n \)
   (b) \( t\cos nt \)
   (c) \( t\sin nt \)
   (d) \( t^n e^{-bt} \)
   (e) \( t^n e^{-bt} \)
   (f) \( t^2 \cos 3t \)
   (g) \( t^3 \sin 2t \)
   (h) \( e^{4t} \cos t \)
   (i) \( e^{-2t} \sin t \)

4. (a) If \( F(p) \) and \( G(p) \) are the Laplace transforms of \( f(x) \) and \( g(x) \), prove that
   \[
   \int_0^\infty F(p) g(p) \, dp = \int_0^\infty f(x) G(x) \, dx,
   \]
   and hence in particular that
   \[
   \int_0^\infty f(x) \frac{dx}{x} = \int_0^\infty F(p) \, dp.
   \]

   (b) Hence evaluate the following integrals.
   (i) \( \int_0^\infty \frac{\sin ax}{x} \, dx \), \( a > 0 \).
   (ii) \( \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx \).
5 Solve the following Volterra integral equations.
(a) \[ g(x) = \sin x + \int_0^x \sin(x-u)g(u) \, du. \]
(b) \[ g(x) = e^x + 2 \int_0^x \cos(x-t)g(t) \, dt. \]

6 Solve the following integro-differential equations.
(a) \[ \int_0^x y(u) \, du - y'(x) = x \]
(b) \[ f'(x) - k^2 \int_0^x f(t) \cos k(x-t) \, dt = 0. \]

7 Use Laplace transforms to show that
\[ y = \frac{1}{k} \int_0^x f(u) \sin k(x-u) \, du \]
is a solution of
\[ \frac{d^2y}{dx^2} + k^2y = f(x), \]
when \( y(0) = y'(0) = 0. \)

8 By taking the Laplace transform of the integral
\[ \int_0^x u^{m-1}(x-u)^{n-1} \, du, \]
using the convolution theorem, and then letting \( x = 1 \), show that
\[ B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \]

9 By taking transforms with respect to the parameter \( t \), evaluating the resultant integral and then inverting, show that
(a) \[ \int_{-\infty}^{\infty} \frac{\sin tx^2}{x} \, dx = \int_0^\infty \frac{1 - \cos 2tx}{x^2} \, dx = \pi t, \]
and
(b) \[ \int_0^\infty e^{-tx^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{t}}. \]

5.7 SYSTEMS OF EQUATIONS

Laplace transforms can be used to transform a set of linear system of differential equations with initial conditions into a set of algebraic equations.

Example

Use Laplace transforms to solve the following initial value problem:
\[ \frac{d^2x}{dt^2} = -10x + 4y \]
\[ \frac{d^2y}{dt^2} = 4x - 4y \]
subject to \( x = 0, \ y = 0, \ \frac{dx}{dt} = 1 \) and \( \frac{dy}{dt} = -1 \) when \( t = 0 \).

Note There are two independent variables \( x, y \) and \( t \) is the dependent variable. Therefore, we take Laplace transforms with respect to the dependent variable, \( t \).

Let \( \mathcal{L}\{x(t)\} = X(p) \) and \( \mathcal{L}\{y(t)\} = Y(p) \). Hence, the system of differential equations becomes a system of algebraic equations, namely,
\[ p^2X - px(0) - x'(0) = -10X + 4Y \]
\[ p^2Y - py(0) - y'(0) = 4X - 4Y \]

where \( x(0) = 0, \ x'(0) = 1, \ y(0) = 0 \) and \( y'(0) = -1 \).

Upon substituting the initial conditions, the algebraic equations reduce to
\[
(p^2 + 10)X - 4Y = 1 \\
(p^2 + 4)Y - 4X = -1
\]

*Note* the function of \( p \) is dropped here purely for convenience.

Solving these equations simultaneously, we find that
\[
X(p) = \frac{p^2}{(p^2 + 2)(p^2 + 12)} \\
Y(p) = -\frac{p^2 + 6}{(p^2 + 2)(p^2 + 12)}
\]

We need now find \( x(t) = \mathcal{L}^{-1}(X(p)) \) and \( y(t) = \mathcal{L}^{-1}(Y(p)) \). To do this we have to use partial fractions. Therefore,
\[
X(p) = -\frac{\frac{1}{5}}{p^2 + 2} + \frac{\frac{6}{5}}{p^2 + 12} \\
Y(p) = -\frac{\frac{2}{5}}{p^2 + 2} - \frac{\frac{3}{5}}{p^2 + 12}
\]

Hence,
\[
x(t) = \mathcal{L}^{-1}(X(p)) = \mathcal{L}^{-1}\left\{-\frac{\frac{1}{5}}{p^2 + 2} + \frac{\frac{6}{5}}{p^2 + 12}\right\} \\
= -\frac{\sqrt{2}}{10} \sin \sqrt{2}t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}t
\]

and
\[
y(t) = \mathcal{L}^{-1}(Y(p)) = \mathcal{L}^{-1}\left\{-\frac{\frac{2}{5}}{p^2 + 2} - \frac{\frac{3}{5}}{p^2 + 12}\right\} \\
= -\frac{\sqrt{2}}{5} \sin \sqrt{2}t + \frac{\sqrt{3}}{10} \sin 2\sqrt{3}t
\]

The general solution to the initial value problem can be now written in the following form:
\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} = \begin{pmatrix}
  -\frac{\sqrt{2}}{10} \sin \sqrt{2}t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}t \\
  -\frac{\sqrt{2}}{5} \sin \sqrt{2}t + \frac{\sqrt{3}}{10} \sin 2\sqrt{3}t
\end{pmatrix}.
\]
Exercise 5E

1 Use Laplace transforms to solve the sets of simultaneous differential equations:
   (a) \( \frac{dx}{dt} + 2x + 3y = 0; \quad \frac{dy}{dt} - 3x + 2y = 0; \)
       \( x = 1, y = 0 \) when \( t = 0 \)
   (b) \( \frac{dy}{dt} - 2x + y = 0; \quad 2\frac{dx}{dt} + y = 4\cos t; \)
       \( y = 0, x = 2 \) when \( t = 0 \)

2 Use Laplace transforms to solve the following initial value problem.
   (a) \( \frac{dx}{dt} = -x + y \)
       \( \frac{dy}{dt} = 2x \)
       where \( x(0) = 0 \) and \( y(0) = 1 \).
   (b) \( \frac{dx}{dt} = 2y + e^t \)
       \( \frac{dy}{dt} = 8x - t \)
       \( x(0) = 1 \) and \( y(0) = 1 \).

   (c) \( \frac{d^2x}{dt^2} + x - y = 0 \)
       \( \frac{d^2y}{dt^2} + y - x = 0 \)
       where \( x(0) = 0, x'(0) = -2 \)
   and
   \( y(0) = 1, y'(0) = 1 \).

   (d) \( \frac{dx}{dt} = 4x - 2y + 2h(t - 1) \)
       \( \frac{dy}{dt} = 3x - y + h(t - 1) \)
       where \( x(0) = 0 \) and \( y(0) = \frac{1}{2} \).
### 5.8 TABLE OF LAPLACE TRANSFORMS

\[
x(t) \quad \quad \quad \quad \quad \quad \quad X(p) = \int_{0}^{\infty} x(t) e^{-pt} dt
\]

<table>
<thead>
<tr>
<th>(x(t))</th>
<th>(X(p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{p})</td>
</tr>
<tr>
<td>(t^n, \ n = 0, 1, 2, \ldots)</td>
<td>(\frac{n!}{p^{n+1}})</td>
</tr>
<tr>
<td>(t^r, \ r &gt; -1)</td>
<td>(\frac{\Gamma(r + 1)}{p^{r+1}})</td>
</tr>
<tr>
<td>(e^{-bt})</td>
<td>(\frac{1}{p + b})</td>
</tr>
<tr>
<td>(\sin nt)</td>
<td>(\frac{n}{p^2 + n^2})</td>
</tr>
<tr>
<td>(\cos nt)</td>
<td>(\frac{p}{p^2 + n^2})</td>
</tr>
<tr>
<td>(\sinh nt)</td>
<td>(\frac{n}{p^2 - n^2})</td>
</tr>
<tr>
<td>(\cosh nt)</td>
<td>(\frac{p}{p^2 - n^2})</td>
</tr>
<tr>
<td>(e^{-bt} f(t))</td>
<td>(F(p + b))</td>
</tr>
<tr>
<td>(h(t - a))</td>
<td>(\frac{e^{-ap}}{p})</td>
</tr>
<tr>
<td>(f(t - a)h(t - a))</td>
<td>(e^{-ap} F(p))</td>
</tr>
<tr>
<td>(\delta(t - a))</td>
<td>(e^{-ap})</td>
</tr>
<tr>
<td>(\delta(t - a)f(t))</td>
<td>(e^{-ap} F(p - a))</td>
</tr>
</tbody>
</table>
\[
\begin{align*}
x(t) & \quad X(p) = \int_0^\infty x(t) e^{-pt} \, dt \\
& \quad f'(t) \quad \quad p F(p) - f(0) \\
& \quad f''(t) \quad \quad p^2 F(s) - p f(0) - f'(0) \\
& \quad \int_0^t f(u) \, du \quad \quad \frac{1}{p} F(p) \\
& \quad \int_0^t f(t - u) g(u) \, du \quad \quad F(p) G(p) \\
& \quad t f(t) \quad \quad \quad \quad - \frac{d}{dp} F(p) \\
& \quad t \sin nt \quad \quad \quad \quad \frac{2pn}{(p^2 + n^2)^2} \\
& \quad t \cos nt \quad \quad \quad \quad \frac{p^2 - n^2}{(p^2 + n^2)^2} \\
& \quad \sin nt - nt \cos nt \quad \quad \quad \quad \frac{2n^3}{(p^2 + n^2)^2} \\
& \quad \sin(at + b) \quad \quad \frac{p \sin b + a \cos p}{(p^2 + a^2)} \\
& \quad \cos(at + b) \quad \quad \frac{p \cos b - a \sin b}{(p^2 + a^2)} \\
& \quad J_0(x) \quad \quad \frac{1}{\sqrt{p^2 + 1}}
\end{align*}
\]