Basic Number Properties: Associative, Commutative, and Distributive

Basic Number Properties: Associative, Commutative, and Distributive (page 1 of 2)

There are three basic properties of numbers, and your textbook will probably have just a little section on these properties, somewhere near the beginning of the course, and then you'll probably never see them again (until the beginning of the next course). My impression is that covering these properties is a holdover from the "New Math" fiasco of the 1960s. While the topic will start to become relevant in matrix algebra and calculus (and become amazingly important in advanced math, a couple years after calculus), they really don't matter a whole lot now.

Why not? Because every math system you've ever worked with has obeyed these properties! You have never dealt with a system where \( a \times b \) did not in fact equal \( b \times a \), for instance, or where \( (a \times b) \times c \) did not equal \( a \times (b \times c) \). Which is why the properties probably seem somewhat pointless to you. Don't worry about their "relevance" for now; just make sure you can keep the properties straight so you can pass the next test. The lesson below explains how I kept track of the properties.

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Distributive Property

The Distributive Property is easy to remember, if you recall that "multiplication distributes over addition". Formally, they write this property as "\( a(b + c) = ab + ac \)". In numbers, this means, that \( 2(3 + 4) = 2 \times 3 + 2 \times 4 \). Any time they refer in a problem to using the Distributive Property, they want you to take something through the parentheses (or factor something out); any time a computation depends on multiplying through a parentheses (or factoring something out), they want you to say that the computation used the Distributive Property.

- **Why is the following true?** \( 2(x + y) = 2x + 2y \)

Since they distributed through the parentheses, this is true by the Distributive
Basic Number Properties: Associative, Commutative, and Distributive

- Use the Distributive Property to rearrange: $4x - 8$

The Distributive Property either takes something through a parentheses or else factors something out. Since there aren't any parentheses to go into, you must need to factor out of. Then the answer is "By the Distributive Property, $4x - 8 = 4(x - 2)$"

"But wait!" you say. "The Distributive Property says multiplication distributes over addition, not subtraction!" What gives?" You make a good point. This is one of those times when it’s best to be flexible. You can either view the contents of the parentheses as the subtraction of a positive number ($"x - 2"$) or else as the addition of a negative number ($"x + (-2)"$). In the latter case, it’s easy to see that the Distributive Property applies, because you’re still adding; you're just adding a negative.

The other two properties come in two versions each: one for addition and the other for multiplication. (Note that the Distributive Property refers to both addition and multiplication, too, but to both within just one rule.)

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**Associative Property**

The word "associative" comes from "associate" or "group"; the Associative Property is the rule that refers to grouping. For addition, the rule is "$a + (b + c) = (a + b) + c$"; in numbers, this means

$2 + (3 + 4) = (2 + 3) + 4$. For multiplication, the rule is "$a(bc) = (ab)c$"; in numbers, this means $2(3\times4) = (2\times3)4$. Any time they refer to the Associative Property, they want you to regroup things; any time a computation depends on things being regrouped, they want you to say that the computation uses the Associative Property.

- **Rearrange, using the Associative Property:** $2(3x)$

  They want you to regroup things, not simplify things. In other words, they do not want you to say "$6x$". They want to see the following regrouping: $(2\times3)x$

- **Simplify $2(3x)$, and justify your steps.**
In this case, they do want you to simplify, but you have to tell why it's okay to do... just exactly what you've always done. Here's how this works:

<table>
<thead>
<tr>
<th>2(3x)</th>
<th>original (given) statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2×3)x</td>
<td>by the Associative Property</td>
</tr>
<tr>
<td>6x</td>
<td>simplification (2×3 = 6)</td>
</tr>
</tbody>
</table>

- Why is it true that 2(3x) = (2×3)x?

Since all they did was regroup things, this is true by the Associative Property.

Commutative Property

The word "commutative" comes from "commute" or "move around", so the Commutative Property is the one that refers to moving stuff around. For addition, the rule is "a + b = b + a"; in numbers, this means 2 + 3 = 3 + 2. For multiplication, the rule is "ab = ba"; in numbers, this means 2×3 = 3×2. Any time they refer to the Commutative Property, they want you to move stuff around; any time a computation depends on moving stuff around, they want you to say that the computation uses the Commutative Property.

- Use the Commutative Property to restate "3×4×x" in at least two ways.

They want you to move stuff around, not simplify. In other words, the answer is not "12x"; the answer is any two of the following:

4 × 3 × x, 4 × x × 3, 3 × x × 4, x × 3 × 4, and x × 4 × 3

- Why is it true that 3(4x) = (4x)(3)?

Since all they did was move stuff around (they didn't regroup), this is true by the Commutative Property.

Worked examples
• Simplify $3a - 5b + 7a$. Justify your steps.

I'm going to do the exact same algebra I've always done, but now I have to give the name of the property that says its okay for me to take each step. The answer looks like this:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3a - 5b + 7a$</td>
<td>original (given) statement</td>
</tr>
<tr>
<td>$3a + 7a - 5b$</td>
<td>Commutative Property</td>
</tr>
<tr>
<td>$(3a + 7a) - 5b$</td>
<td>Associative Property</td>
</tr>
<tr>
<td>$a(3 + 7) - 5b$</td>
<td>Distributive Property</td>
</tr>
<tr>
<td>$a(10) - 5b$</td>
<td>simplification ($3 + 7 = 10$)</td>
</tr>
<tr>
<td>$10a - 5b$</td>
<td>Commutative Property</td>
</tr>
</tbody>
</table>

The only fiddly part was moving the "$- 5b$" from the middle of the expression (in the first line of the table above) to the end of the expression (in the second line). If you need help keeping your negatives straight, convert the "$- 5b$" to "$+ (-5b)$". Just don't lose that minus sign!

• Simplify $23 + 5x + 7y - x - y - 27$. Justify your steps.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$23 + 5x + 7y - x - y - 27$</td>
<td>original (given) statement</td>
</tr>
<tr>
<td>$23 - 27 + 5x - x + 7y - y$</td>
<td>Commutative Property</td>
</tr>
<tr>
<td>$(23 - 27) + (5x - x) + (7y - y)$</td>
<td>Associative Property</td>
</tr>
<tr>
<td>$(-4) + (5x - x) + (7y - y)$</td>
<td>simplification ($23 - 27 = -4$)</td>
</tr>
<tr>
<td>$(-4) + x(5 - 1) + y(7 - 1)$</td>
<td>Distributive Property</td>
</tr>
<tr>
<td>$-4 + x(4) + y(6)$</td>
<td>simplification</td>
</tr>
<tr>
<td>$-4 + 4x + 6y$</td>
<td>Commutative Property</td>
</tr>
</tbody>
</table>

• Simplify $3(x + 2) - 4x$. Justify your steps.

<table>
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<tr>
<th>Expression</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$3(x + 2) - 4x$</td>
<td>original (given) statement</td>
</tr>
<tr>
<td>$3x + 3\times 2 - 4x$</td>
<td>Distributive Property</td>
</tr>
<tr>
<td>$3x + 6 - 4x$</td>
<td>simplification ($3\times 2 = 6$)</td>
</tr>
<tr>
<td>$3x - 4x + 6$</td>
<td>Commutative Property</td>
</tr>
<tr>
<td>$(3x - 4x) + 6$</td>
<td>Associative Property</td>
</tr>
<tr>
<td>$x(3 - 4) + 6$</td>
<td>Distributive Property</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>$x(-1) + 6$</th>
<th>simplification $(3 - 4 = -1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-x + 6$</td>
<td>Commutative Property</td>
</tr>
</tbody>
</table>

- **Why is it true that** $3(4 + x) = 3(x + 4)$?

  All they did was move stuff around: **Commutative Property**

- **Why is** $3(4x) = (3 \times 4)x$?

  All they did was regroup: **Associative Property**

- **Why is** $12 - 3x = 3(4 - x)$?

  They factored: **Distributive Property**


**Other Number Properties: Identities, Inverses, Symmetry, etc. (page 2 of 2)**

If your textbook gets really ornate, you may have to delve into some of the more esoteric properties of numbers. For this, you need to know that "the identity" is whatever doesn't change your number at all, and "the inverse" is whatever turns your number into the identity.

For addition, "the identity" is zero, because adding zero to anything doesn't change anything. The "inverse" is the additive inverse: it's the same number, but with the opposite sign. For instance, suppose your number is $-6$, and you're adding. The identity is zero, and the inverse is 6, because $-6 + 6 = 0$.

For multiplication, "the identity" is one, because multiplying by one doesn't change anything. The "inverse" is the multiplicative inverse: the same number, but on the opposite
side of the fraction line. For instance, suppose your number is \(-6\), and you're multiplying. The identity is one, and the inverse is \(\frac{-1}{6}\), because \((-6)(\frac{-1}{6}) = 1\).

You also know (if you've done any equation solving) that you can do anything you want to an equation, as long as you do the same thing to both sides. This is the "property of equality".

The basic fact that you need for solving many equations, especially quadratics, is that, if \(p \times q = 0\), then must have either \(p = 0\) or else \(q = 0\). The only way you can multiply two things and end up with zero is if one (or both) of those two things was zero to start with. This is the "zero-product property".

And there are some properties that you use to solve word problems, especially where substitution is required. Anything equals itself: this is the "reflexive" (reflecting onto itself) property. Also, it doesn't matter which order the equality is in; if \(x = y\), then \(y = x\): this is the "symmetric" (they match) property. You can "cut out the middleman", so to speak; if \(x = y\) and \(y = z\), then you can say that \(x = z\): this is the "transitive" (moving across) property. Two numbers are either equal to each other or unequal; this is the "trichotomy" law (so called because there are three cases for two given numbers, \(a < b\), \(a = b\), or \(a > b\)). And you can plug in for variables, so if \(x = 3\), then \(4x = 12\), because \(4x = 4(3)\): this is the "substitution" property.

Here are some examples. Note: textbooks vary somewhat in the names they give these properties; you'll need to refer to the examples in your book to know the exact format you should use.

**Determine which property was used.**

- \(1 \times 7 = 7\)
  
  They multiplied, and they didn't change anything: the **multiplicative identity**.

- \(-7y = -7y\)
  
  This is obvious: anything equals itself. They used the **reflexive property**.
If 10 = y, then y = 10.

When solving an equation, I might rearrange things so I end up with the variable on the left. But I only switched sides; I didn't actually change anything: the symmetric property.

x + 0 = x

They added, and they didn't change anything: the additive identity.

If 2(a + b) = 3c, and a + b = 9, then 2(9) = 3c.

You might be torn here between the transitive property and the substitution property. If you look closely, what they did was substitute "9" for "a + b", so they used the substitution property.

2 = x, so 2 + 5 = x + 5

They did the backwards of solving an equation, but the point is that they were working with an equation. They changed the equation by adding equal things to both sides: the additive property of equality. Copyright © Elizabeth Stapel 2000-2011 All Rights Reserved

If x + 2 = 10, then x + 2 + (−2) equals what, and why?

They solved the equation by getting rid of the 2 from both sides. Since they added the same thing to both sides, they got x = 8 by the additive property of equality.

(x − 3)(x + 4) = 0, so x = 3 or x = −4.

They set the quadratic equal to zero, factored, and then solved each factor: the zero-product property.

4x = 8, so x = 2

They solved the equation by dividing both sides by 4, or, which is the same thing, multiplying both sides by \( \frac{1}{4} \). In other words, they changed the equation by do-
ing the same multiplying to both sides: the multiplicative property of equality.

- **If \( x \) is not equal to \( y \) and not less than \( y \), what must be true of \( x \), and why?**

By the trichotomy law, there are only three possible relationships between \( x \) and \( y \), and they’ve eliminated two of them. Then \( x > y \), by the trichotomy law.

- **\( x + (-x) = 0 \)**

They added, and they ended up with zero: the additive inverse.

- **\((3/3)(2/5) + (5/5)(4/3) = 6/15 + 20/15\)**

They converted to a common denominator by multiplying both fractions by a useful form of 1; remember that \(3/3\) and \(5/5\) are just 1! So they used the multiplicative identity.

- **If \( 5x = 0 \), what is \( x \), and why?**

You can do this in either of two ways: multiply both sides by \(1/5\) (the multiplicative property of equality) and then get \( x = 0 \), or you could say that, since 5 doesn't equal zero, then \( x \) must equal zero (by the zero-product property).

- **\((2/3)(3/2) = 1\)**

They multiplied, and they ended up with one: the multiplicative inverse.

- **If \( 3x + 2 = y \) and \( y = 8 \), then \( 3x + 2 = 8 \).**

You might be torn here between the transitive property and the substitution property. What they did here was "cut out the middleman" by removing the "\( y \)" in the middle, so they used the transitive property.

- **If \( -x = 14 \), what does \( x \) equal, and why?**

To solve this, you would multiply both sides by a negative one, to cancel off the minus sign. So:
\[ x = -14, \text{ by the multiplicative property of equality.} \]

- If \( x = 3 \) and \( y = -4 \), then what does \( xy \) equal, and why?

By substitution (plugging in for the variables), you get \((3)(-4)\). In other words:

\[ xy = -12, \text{ by the substitution property.} \]

- Can \( x < x \)? Why or why not?

By the reflexive property, \( x = x \). By the trichotomy law, if \( a = b \) then \( a \) cannot be less than \( b \). So the answer is "no, by the reflexive property and the trichotomy law".

Don't let the seeming pointlessness of these questions bother you. Instead, view this stuff as "gimme" questions for the next test.