Factorials: The definition of $n!$ (read “$n$ factorial”) is

$$n! = n(n - 1)(n - 2)(n - 3) \cdots 3 \cdot 2 \cdot 1$$

For example $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. We will use the standard convention $0! = 1$ [I really want to emphasize this point, but I'm afraid using exclamation points would just make things more confusing... so REMEMBER: $0! = 1$.]

Notice $1052! = 1052 \cdot 1051!$. Similarly $254! = 254 \cdot 253 \cdot 252 \cdot 251!$. Because of this we see

$$\frac{102!}{99!} = \frac{102 \cdot 101 \cdot 100 \cdot 99!}{99!} = 102 \cdot 101 \cdot 100,$$

or in general

$$\frac{n!}{r!} = n(n - 1)(n - 2) \cdots (r + 2)(r + 1)$$

whenever $r < n$.

**Example 1.** Let $f(x) = \ln(x)$. Let’s look at the first few derivatives:

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{(-2)(-1)}{x^3} = \frac{2}{x^3}$$

$$f''''(x) = \frac{(-3)(-2)(-1)}{x^4} = \frac{-6}{x^4}$$

$$f^{(5)}(x) = \frac{(-4)(-3)(-2)(-1)}{x^5} = \frac{24}{x^5}$$

... 

$$f^{(n)}(x) = \frac{(-(n - 1))(-(n - 2)) \cdots (-3)(-2)(-1)}{x^n} = \frac{(-1)^{n-1}(n - 1)!}{x^n}$$

So

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n - 1)!}{x^n}$$

Notice that when $n = 1$ this formula gives

$$f'(x) = \frac{(-1)^0(0)!}{x} = \frac{1}{x}$$

because $0! = 1$. 
Example 2. Let \( f(x) = xe^x \). We will begin by finding the first derivative using the product rule:

\[
    f'(x) = \frac{d}{dx}(xe^x) = e^x + xe^x = e^x + f(x)
\]

So \( f'(x) = e^x + f(x) \). Notice we have found a formula for \( f'(x) \) which involves \( f(x) \). We will use this to discover a pattern in the first few derivatives:

\[
    f''(x) = \frac{d}{dx}(e^x + f(x)) = \frac{d}{dx}(e^x) + f'(x) = e^x + (e^x + f(x)) = 2e^x + f(x)
\]

\[
    f'''(x) = \frac{d}{dx}(2e^x + f(x)) = \frac{d}{dx}(2e^x) + f'(x) = 2e^x + (e^x + f(x)) = 3e^x + f(x)
\]

\[
    f''''(x) = \frac{d}{dx}(3e^x + f(x)) = \frac{d}{dx}(3e^x) + f'(x) = 3e^x + (e^x + f(x)) = 4e^x + f(x)
\]

\[
    \vdots
\]

\[
    f^{(n)}(x) = \frac{d}{dx}((n-1)e^x + f(x)) = \frac{d}{dx}((n-1)e^x) + f'(x) = (n-1)e^x + (e^x + f(x)) = ne^x + f(x)
\]

Since \( f(x) = xe^x \) we see \( f^{(n)}(x) = ne^x + xe^x \).

Example 3. Let \( f(x) = \sin(x) \). In class we noticed that

\[
    f'(x) = \cos(x)
\]

\[
    f''(x) = -\sin(x)
\]

\[
    f'''(x) = -\cos(x)
\]

\[
    f''''(x) = \sin(x)
\]

So that \( f''''(x) = f(x) \), thus if we listed higher derivatives, we would be repeating the four expressions \( \cos(x) \), \( -\sin(x) \), \( -\cos(x) \), \( \sin(x) \) over and over again.

Now remember from trigonometry \( \sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a) \), thus

\[
    \sin \left( \frac{\pi}{2} + x \right) = \cos(x)
\]

\[
    \sin \left( \frac{2\pi}{2} + x \right) = -\sin(x)
\]

\[
    \sin \left( \frac{3\pi}{2} + x \right) = -\cos(x)
\]

\[
    \sin \left( \frac{4\pi}{2} + x \right) = \sin(x)
\]

\[
    \vdots
\]

Thus we can write the \( n \)th derivative as follows:

\[
    f^{(n)}(x) = \sin \left( \frac{n\pi}{2} + x \right).
\]
Example 4. Let $f(x) = x \sin(x)$. If you find the first few derivatives, you should get:

$$f'(x) = x \cos(x) + \sin(x)$$
$$f''(x) = -x \sin(x) + 2 \cos(x)$$
$$f'''(x) = -x \cos(x) - 3 \sin(x)$$
$$f''''(x) = x \sin(x) - 4 \cos(x)$$
$$f^{(5)}(x) = x \cos(x) + 5 \sin(x)$$
$$f^{(6)}(x) = -x \sin(x) + 6 \cos(x)$$
$$f^{(7)}(x) = -x \cos(x) - 7 \sin(x)$$
$$f^{(8)}(x) = x \sin(x) - 8 \cos(x)$$
$$f^{(9)}(x) = x \cos(x) + 9 \sin(x)$$

\[\vdots\]

You should convince yourself (using addition identities for sine and cosine) that the following formula gives the $n$th derivative:

$$f^{(n)}(x) = x \sin\left(\frac{n\pi}{2} + x\right) - n \cos\left(\frac{n\pi}{2} + x\right)$$

Exercises: Here are a few exercises on $n$th derivatives which might be fun for you to do. I won’t be collecting them for credit, but I will be happy to look over your solutions.

1. Find a formula for the $n$th derivative of the following functions.

   (a) $f(x) = \cos(x)$ (Hint: look at example 3)
   (b) $f(x) = x \cos(x)$ (Hint: look at example 4)
   (c) $f(x) = x \ln(x)$ for $n > 1$ (Hint: use example 1)
   (d) $f(x) = e^{5x}$
   (e) $f(x) = x^n$
   (f) $f(x) = x^k$ where $k$ is some integer greater than $n$.

2. Let

   $$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

   be an arbitrary polynomial of degree $n$. [i.e. the $a_i$’s are constants]

   (a) Find a formula for $f^{(m)}(x)$ where $m > n$.
   (b) Find a formula for $f^{(m)}(x)$ where $m = n$.
   (c) Find a formula for $f^{(m)}(x)$ where $m < n$. (this one is harder)